Find the linear basis functions for the triangle $K$ with vertices at $(0,0)$, $(h,0)$ and $(0,h)$. Show that the corresponding element stiffness matrix is given by
$$
\begin{bmatrix}
1 & -\frac{1}{2} & -\frac{1}{2} \\
-\frac{1}{2} & \frac{1}{2} & 0 \\
-\frac{1}{2} & 0 & \frac{1}{2}
\end{bmatrix}
$$

Using this result, show that the linear system (1.25) of Example 1.1. has the stated form (pages 30-31) (there is a typo in the textbook regarding the matrix)

Let $V$ be a Hilbert space and $L : V \to \mathbb{R}$ be a linear form. Show that $L$ is bounded if and only if $L$ is continuous. Then directly apply this result to a bilinear form $a : V \times V \to \mathbb{R}$.

Let $V$ be a Hilbert space and the (nonlinear) operator $A : V \to V$, satisfying

(i) there is $M \geq 0$ s.t. $\forall u, v \in V, \|Au - Av\| \leq M\|u - v\|$  
(ii) there is $\alpha > 0$ s.t. $\forall u, v \in V, \langle Au - Av, u - v \rangle \geq \alpha\|u - v\|^2$

Show that the nonlinear equation $Au = f$ has a unique solution (for some $f \in V$), using the Banach fixed point theorem and the same technique for proving the Lax-Milgram theorem (introduce the function $g_\lambda$).

Let $V$ be a complex Hilbert space, $a : V \times V \to \mathbb{C}$ a sesquilinear form, $L : V \to \mathbb{C}$ an anti-linear form. Assume that $a$ is Hermitian, thus $a(v,u) = \overline{a(u,v)}$, $\forall u, v \in V$, and that $a(v,v) \geq 0$. Consider the problems

(V) Find $u \in V$ s.t. $a(u,v) = L(v), \forall v \in V$,

(M) Find $u \in V$ s.t. $J(u) = \inf_{v \in V} J(v)$,

with $J : V \to \mathbb{R}$ defined by $J(v) = \frac{1}{2}a(v,v) - \text{Re}L(v)$.

Show that $u \in V$ is solution of (V) iff $u \in V$ is solution of (M).

• Note: Over the complex space $C$, with the above notations, we say that $a : V \times V \to C$ is a sesquilinear form if $a$ is linear in the first argument (under the usual addition and scalar multiplication) and antilinear in the second argument (usual addition but $a(u, \lambda v) = \overline{\lambda}a(u, v)$). Note that if $\text{Re}a(v,v) \geq \alpha\|v\|^2$, together with $a$ and $L$ bounded, then the Lax-Milgram Lemma holds in the complex case with the same proof.