269C HW#2, due on Wednesday, May 4

[1] Find the linear basis functions for the triangle K with vertices at $(0,0)$, $(h,0)$ and $(0,h)$. Show that the corresponding element stiffness matrix is given by

$$\begin{bmatrix} 1 & -\frac{1}{2} & -\frac{1}{2} \\ -\frac{1}{2} & \frac{1}{2} & 0 \\ -\frac{1}{2} & 0 & \frac{1}{2} \end{bmatrix}$$

Using this result, show that the linear system (1.25) of Example 1.1. has the stated form (pages 30-31) (there is a typo in the textbook regarding the matrix)

[2] (a) Give a weak variational formulation of the problem

$$\frac{d^4u}{dx^4} = f \quad \text{for} \quad 0 < x < 1,$$
$$u(0) = u''(0) = u'(1) = u'''(1) = 0,$$

and show that the assumptions of the Lax-Milgram Lemma are satisfied. Which boundary conditions are essential and which are natural ?

(b) Solve the same problem with the following alternative boundary conditions:

$$u(0) = -u''(0) + \gamma u'(0) = 0, \quad u(1) = u''(1) + \gamma u'(1) = 0,$$

where γ is a positive constant.

[3] Give a weak variational formulation of the Robin's problem

$$-\triangle u = f \quad \text{in } \Omega, \quad \gamma u + \frac{\partial u}{\partial n} = g \text{ on } \Gamma,$$

where γ is a constant. When are the assumptions of the Lax-Milgram Lemma satisfied ?

[4] Consider the Neumann problem

$$-\triangle u = f \text{ in } \Omega,$$
$$\frac{\partial u}{\partial n} = g \text{ on } \Gamma = \partial\Omega,$$
$$\int_\Omega u(x)dx = 0.$$

where $f : \Omega \to R$ and $g : \partial\Omega \to R$ satisfy the *compatibility condition*

$$\int_\Omega f(x)dx + \int_{\partial\Omega} g(x)d\sigma(x) = 0.$$

(a) Why condition "$\int_\Omega u(x)dx = 0$" was added here ? Why do we need the compatibility condition ?

(b) Give a weak variational formulation of the problem, and prove that the conditions of the Lax-Milgram Lemma are satisfied, under the necessary assumptions on f and g that you would specify.