HW #1. Math 269C (due on Wednesday, April 20)
(late homework accepted)
(please read note on the 2nd page)

[1] Show that if w is continuous on $[0,1]$, and

$$\int_0^1 wvdx = 0, \text{ for all } v \in V,$$

with $V = \{v : [0,1] \to R,\ \textit{continuous},\ v(0) = v(1) = 0,\ v' \textit{ piecewise} - \textit{continuous and bounded}\}$, then $w(x) = 0$ for $x \in [0,1]$.

[2] Construct a finite-dimensional subspace V_h of V (from problem [1]) consisting of functions which are quadratic on each subinterval I_j of a partition of $I = (0,1)$. How can one choose the parameters to describe such functions ? Find the corresponding basis functions. Then formulate a finite element method for (D) using the space V_h and write down the corresponding linear system of equations in the case of a uniform partition. Recall that (D) is

$$-u'' = f \text{ in } (0,1), \quad u(0) = u(1) = 0.$$

[3] Consider the BVP

$$\frac{d^4u}{dx^4} = f, \quad 0 < x < 1, \quad u(0) = u'(0) = u(1) = u'(1) = 0.$$

(a) Show that this problem can be given the following variational formulation: Find $u \in W$ such that

$$(u'', v'') = (f, v), \quad \forall v \in W,$$

where $W = \{v : v \text{ and } v' \text{ are continuous on } [0,1],\ v'' \text{ is piecewise-continuous and } v(0) = v'(0) = v(1) = v'(1) = 0\}$.

(b) For $I = [a,b]$ an interval, define

$$P_3(I) = \{v :\ v \text{ is a polynomial of degree } \leq 3 \text{ on } I\},$$

i.e. v has the form $v(x) = a_3x^3 + a_2x^2 + a_1x + a_0$ for $x \in I$, and $a_i \in R$.

Show that $v \in P_3(I)$ is uniquely determined by the values $v(a)$, $v'(a)$, $v(b)$, $v'(b)$. Find the corresponding basis functions (the basis function corresponding to the value $v(a)$ is the cubic polynomial v such that $v(a) = 1$, $v'(a) = 0$, $v(b) = 0$, $v'(b) = 0$, etc.)

(c) Starting from (b) construct a finite-dimensional subspace W_h of W consisting of piecewise-cubic functions. Specify suitable parameters to describe the functions in W_h and determine the corresponding basis functions.

(d) Formulate a FEM for the problem based on the space W_h. Find the corresponding linear system of equations in the case of a uniform partition.

• Note: for the question "Find the corresponding basis functions." in problems [2], [3], feel free to use some software that can evaluate the integrals for you (it is not difficult but tedious and not so interesting). Feel free to leave out this part of the question, or just give the formulas without completely evaluating the integrals.