269C: HW#5 (final assignment) You can leave the final assignment in my mailbox, or with Babette Dalton in MS 7619 between 7-3pm, or you can slide it under the door of my office MS 7620-D.

Tentative due date: June 15, 2015.

[**1**] Consider the PDE (in the distributional sense)

$$-\triangle u + k^2 u = f \quad \text{in } R^n,$$

where k is a constant. Let $s \in R$. Show that, for all $f \in H^s(R^n)$, there exists a unique $u \in H^{s+2}(R^n)$, solution of the PDE, with $k \in R$, $k \neq 0$.

Hint: use the Fourier transform.

[**2**] Let $I = [0, h]$ and let $\pi v \in P_1(I)$ be the linear interpolant that agrees with $v \in C^2(I)$ at the end points of I. Using the technique of the proof of Thm. 4.1, prove estimates for $\|v - \pi v\|_{L^\infty(I)}$ and $\|v' - (\pi v)'\|_{L^\infty(I)}$, cf. (1.12) and (1.13).

[**3**] Using the general results from Chapter 4, estimate the error $\|u - u_h\|_{H^2}$ for Problem 1.5 and Example 2.4.

[**4**] Using polar coordinates (r, θ), let $\Omega = \{(r, \theta) : \ 0 < r < 1, \ 0 < \theta < \omega\}$ be a pie-shaped domain of angle ω.

(i) Prove that the function $u(r, \theta) = r^\gamma \sin(\gamma\theta)$, $\gamma = \frac{\pi}{\omega}$, satisfies: $\triangle u = 0$ in Ω, $u = 0$ on the straight parts of the boundary of Ω.

(ii) Consider the cases $\omega > \pi$ and $\omega < \pi$ and study the condition $u \in H^2(\Omega)$ (see the discusssion in Section 4.5).

[**5**] The following elliptic problem is approximated by the finite element method,

$$\begin{aligned}
-\mathrm{div}\Big(a(x)\nabla u(x)\Big) &= f(x), \ x \in \Omega \subset R^2, \\
u(x) &= 2, \ x \in \partial\Omega_1, \\
\frac{\partial u(x)}{\partial x_1} + u(x) &= 0, \ x \in \partial\Omega_2, \\
\frac{\partial u(x)}{\partial x_2} &= 0, \ x \in \partial\Omega_3,
\end{aligned}$$

where

$$
\begin{aligned}
\Omega &= \{(x_1, x_2): \; 0 < x_1 < 1, \; 0 < x_2 < 1\}, \\
\Gamma_1 = \partial\Omega_1 &= \{(x_1, x_2): \; x_1 = 0, \; 0 \le x_2 \le 1\}, \\
\Gamma_2 = \partial\Omega_2 &= \{(x_1, x_2): \; x_1 = 1, \; 0 \le x_2 \le 1\}, \\
\Gamma_3 = \partial\Omega_3 &= \{(x_1, x_2): \; 0 < x_1 < 1, \; x_2 = 0, \; 1\},
\end{aligned}
$$

and

$$0 < A \le a(x) \le B.$$

(a) Determine an appropriate weak formulation of the problem.

(b) Prove conditions on the corresponding linear and bilinear forms which are needed for existence and uniqueness and for the convergence of a finite element method (assume $f \in L^2(\Omega)$, $a \in L^\infty(\Omega)$).

(c) Describe briefly a finite element mesh, a FEM using P_1 elements, and a set of basis functions such that the linear system from the finite element approximation is sparse and of band structure.

[6] (a) Develop and describe the piecewise linear Galerkin finite element approximation of,

$$
\begin{aligned}
-\nabla \cdot a(x)\nabla u + b(x)u = f(x), &\quad x = (x_1, x_2) \in \Omega, \\
u = 2, &\quad x \in \partial\Omega_1, \\
\frac{\partial u}{\partial x_1} + \frac{\partial u}{\partial x_2} + u = 2, &\quad x \in \partial\Omega_2,
\end{aligned}
$$

where $f \in L^2(\Omega)$,

$\Omega = \{x | \; x_1 > 0, \; x_2 > 0, \; x_1 + x_2 < 1\}$,

$\partial\Omega_1 = \{x | \; x_1 = 0, \; 0 \le x_2 \le 1\} \cup \{x | \; x_2 = 0, \; 0 \le x_1 \le 1\}$,

$\partial\Omega_2 = \{x | \; x_1 > 0, \; x_2 > 0, \; x_1 + x_2 = 1\}$,

$0 < a \le a(x) \le A$, $0 < b \le b(x) \le B$.

(b) Justify the approximation by analyzing the appropriate bilinear and linear forms. Give a convergence estimate and quote the appropriate thoerems for convergence.