HW #4, 269C Due on Monday, June 1st.

[**1**] Recall the "coercive" inhomogeneous Neumann problem:

$$(D)\ \Big\{-\triangle u + u = f \text{ in } \Omega,\ \ \frac{\partial u}{\partial \vec{n}} = g \text{ on } \partial\Omega\Big\}.$$

We know that the corresponding weak variational problem (with $V = H^1(\Omega)$) is: find $u \in H^1(\Omega)$ such that

$$(V)\ \ \int_\Omega (\nabla u \cdot \nabla v + uv)dx = \int_\Omega fv dx + \int_{\partial\Omega} gv ds,$$

for all $v \in H^1(\Omega)$. We also know that $(D) \Rightarrow (V)$. Assume now that u,f, and g are sufficiently smooth. Show that, if u is a smooth solution of (V) (for example $u \in C^2(\overline{\Omega})$, then u satisfies (D); in other words, $(V) \Rightarrow (D)$.

[**2**] Let K be a tetrahedron with vertices a^i, $i = 1, ..., 4$, and let a^{ij} denote the midpoint on the straight line $a^i a^j$, $i < j$. Show that a function $v \in P_2(K)$ is uniquely determined by the degrees of freedom: $v(a^i)$, $v(a^{ij})$, $i, j = 1, ..., 4$, $i < j$. Show that the corresponding finite element space V_h satisfies $V_h \subset C^0(\Omega)$, assuming continuity at all degrees of freedom.

[**3**] Let K be a triangle with vertices a^i, $i = 1, 2, 3$. Suppose that $v \in P_r(K)$ and that v vanishes on the side $a^2 a^3$. Prove that v has the form

$$v(x) = \lambda_1(x) w_{r-1}(x), \quad x \in K,$$

where $w_{r-1} \in P_{r-1}(K)$, and $\lambda_1(x)$ is the affine local basis function corresponding to the node a^1.

[**4**] Let K be a triangle with vertices a^i, $i = 1, 2, 3$, and let a^{ij}, $i < j$, denote the midpoints of the sides of K. Let a^{123} denote the center of gravity of K. Prove that $v \in P_4(K)$ is uniquely determined by the following degrees of freedom

$$\begin{aligned}
&v(a^i),\\
&\frac{\partial v}{\partial x_j}(a^i),\ \ i = 1, 2, 3,\ \ j = 1, 2,\\
&v(a^{ij}),\ \ i, j = 1, 2, 3,\ \ i < j,\\
&v(a^{123}),\ \ \frac{\partial v}{\partial x_j}(a^{123}), j = 1, 2,
\end{aligned}$$

(typo in Exercise 3.8 in the textbook).

Also show that the functions in the corresponding finite element space V_h are continuous, assuming continuity for all degrees of freedom.