HW #1. Math 269C (due on Friday, April 17)

[1] Show that if $w$ is continuous on $[0,1]$, and

$$\int_0^1 w v dx = 0, \text{ for all } v \in V,$$

with $V = \{ v : [0,1] \to \mathbb{R}, \text{ continuous, } v(0) = v(1) = 0, \text{ } v' \text{ piecewise-continuous and bounded} \}$, then $w(x) = 0$ for $x \in [0,1]$.

[2] Construct a finite-dimensional subspace $V_h$ of $V$ (from problem [1]) consisting of functions which are quadratic on each subinterval $I_j$ of a partition of $I = (0,1)$. How can one choose the parameters to describe such functions? Find the corresponding basis functions. Then formulate a finite element method for (D) using the space $V_h$ and write down the corresponding linear system of equations in the case of a uniform partition. Recall that (D) is

$$-u'' = f \text{ in } (0,1), \quad u(0) = u(1) = 0.$$

[3] Consider the BVP

$$\frac{d^4 u}{dx^4} = f, \quad 0 < x < 1, \quad u(0) = u'(0) = u(1) = u'(1) = 0.$$

(a) Show that this problem can be given the following variational formulation: Find $u \in W$ such that

$$(u'', v'') = (f, v), \quad \forall v \in W,$$

where $W = \{ v : v \text{ and } v' \text{ are continuous on } [0,1], \text{ } v'' \text{ is piecewise-continuous and } v(0) = v'(0) = v(1) = v'(1) = 0 \}$.

(b) For $I = [a, b]$ an interval, define

$$P_3(I) = \{ v : v \text{ is a polynomial of degree } \leq 3 \text{ on } I \},$$

i.e. $v$ has the form $v(x) = a_3 x^3 + a_2 x^2 + a_1 x + a_0$ for $x \in I$, and $a_i \in \mathbb{R}$.

Show that $v \in P_3(I)$ is uniquely determined by the values $v(a), \text{ } v'(a), \text{ } v(b), \text{ } v'(b)$. Find the corresponding basis functions (the basis function corresponding to the value $v(a)$ is the cubic polynomial $v$ such that $v(a) = 1, \text{ } v'(a) = 0, \text{ } v(b) = 0, \text{ } v'(b) = 0$, etc.)

(c) Starting from (b) construct a finite-dimensional subspace $W_h$ of $W$ consisting of piecewise-cubic functions. Specify suitable parameters to describe the functions in $W_h$ and determine the corresponding basis functions.

(d) Formulate a FEM for the problem based on the space $W_h$. Find the corresponding linear system of equations in the case of a uniform partition.