Notations:
For \( u \in H^m(\Omega) \), let
\[
\|u\|_{H^m(\Omega)} = \left( \sum_{|\alpha| \leq m} \int_{\Omega} |D^\alpha u(x)|^2 \, dx \right)^{1/2} = \left( \sum_{|\alpha| \leq m} \|D^\alpha u\|^2_{L^2(\Omega)} \right)^{1/2},
\]
\[
|u|_{H^m(\Omega)} = \left( \sum_{|\alpha| = m} \int_{\Omega} |D^\alpha u(x)|^2 \, dx \right)^{1/2} = \left( \sum_{|\alpha| = m} \|D^\alpha u\|^2_{L^2(\Omega)} \right)^{1/2}.
\]

Thm. (Poincaré's Inequality for \( H^1_0(\Omega) \))
Let \( \Omega \) be an open and bounded set in \( \mathbb{R}^n \). Then there is a positive constant \( C = C(\Omega) \) such that, for all \( u \in H^1_0(\Omega) \), we have Poincaré inequality:
\[
\|u\|^2_{L^2(\Omega)} \leq C \|\nabla u\|^2_{L^2(\Omega)}.
\]

Corollary: Let \( m > 0 \) be a positive integer, and let \( \Omega \) be an open and bounded set in \( \mathbb{R}^n \).
Then for \( u \in H^m_0(\Omega) \), we have
\[
\|u\|^2_{L^2(\Omega)} \leq C_m \sum_{|\alpha| = m} \|D^\alpha u\|^2_{L^2(\Omega)},
\]
where \( C_m = \text{constant} \), and \( C \) is the constant from the previous theorem.

Corollary: (same assumptions on \( \Omega \)). \( |u|_{H^m(\Omega)} \) is a norm on \( H^m_0(\Omega) \), equivalent to the norm \( \|u\|_{H^m(\Omega)} \).

Thm. Let \( \Omega \) be a bounded connected open set in \( \mathbb{R}^n \), with sufficiently regular boundary. Then we have for \( u \in H^1(\Omega) \), such that \( \int_{\Omega} u(x) \, dx = 0 \),
\[
\|u\|^2_{L^2(\Omega)} \leq P(\Omega) \|\nabla u\|^2_{L^2(\Omega)}.
\]

More generally, we have for \( u \in H^1(\Omega) \)
\[
\|u\|^2_{L^2(\Omega)} \leq P(\Omega) \|\nabla u\|^2_{L^2(\Omega)} + \frac{1}{|\Omega|} \int_{\Omega} u(x) \, dx \right)^2.
\]

Corollary: \( |u|_{H^1(\Omega)} = \|\nabla u\|_{L^2(\Omega)} \) is a norm equivalent with the norm \( \|u\|_{H^1(\Omega)} \) on the sub-space \( V_0 \) (closed in \( H^1(\Omega) \)) defined by:
\[
V_0 = \{ u \in H^1(\Omega) : \int_{\Omega} u(x) \, dx = 0 \}.
\]

Corollary: Let \( \Omega \) be a bounded connected open set in \( \mathbb{R}^n \), with sufficiently regular boundary \( \Gamma \). Suppose \( \Gamma = \Gamma_1 \cup \Gamma_2 \) with length (area) of \( \Gamma_2 > 0 \). Let
\[
V_{\Gamma_2} = \{ u \in H^1(\Omega) : u|_{\Gamma_2} = 0 \}.
\]
Then \( V_{\Gamma_2} \) is a closed sub-space of \( H^1(\Omega) \) and \( |u|_{H^1(\Omega)} = \|\nabla u\|_{L^2(\Omega)} \) is a norm equivalent with the norm \( \|u\|_{H^1(\Omega)} \) on the sub-space \( V_{\Gamma_2} \).
Remark:
(i) Suppose that $\Omega$ is a bounded connected open set in $\mathbb{R}^n$ which is “very regular” ($\Gamma = \partial \Omega$ is a $n - 1$ dimensional manifold of class $C^\infty$ and $\Omega$ locally on one side of $\Gamma$). For $u \in H^1(\Omega)$, let
\[
\|u\|_{H^1(\Omega),\Gamma}^2 = \|\nabla u\|_{L^2(\Omega)}^2 + \int_{\Gamma} |u|^2 d\Gamma,
\]
where $u|_{\Gamma}$ is the trace of $u$ on $\Gamma$. Then there is a constant $C > 0$ such that
\[
\|u\|_{H^1(\Omega)} \leq C\|u\|_{H^1(\Omega),\Gamma},
\]
for all $u \in H^1(\Omega)$. Therefore, $\|u\|_{H^1(\Omega),\Gamma}$ is a norm equivalent to $\|u\|_{H^1(\Omega)}$ on $H^1(\Omega)$.

(ii) Let $V_{\Gamma} = \left\{ u \in H^1(\Omega), \int_{\Gamma} u d\Gamma = 0 \right\}$. Then $V_{\Gamma}$ is a closed subspace of $H^1(\Omega)$, and $|u|_{H^1(\Omega)}$ is a norm equivalent to $\|u\|_{H^1(\Omega)}$ on $V_{\Gamma}$.

Corollary: Let $\Omega$ an open and bounded domain, with Lipschitz-continuous boundary $\Gamma = \partial \Omega$. Then there is a positive constant $C$ such that
\[
\|u|_{\Gamma}\|_{L^2(\Gamma)} \leq C\|u\|_{H^1(\Omega)}.
\]

Corollary: Over the space $H_0^2(\Omega)$, $\|\Delta u\|_{L^2(\Omega)}$ is a norm, equivalent to $\|u\|_{H^2(\Omega)}$.

• For $s$ a real number, then $u \in H^s(\mathbb{R}^n)$ if
\[
(1 + |\xi|^2)^{s/2} \hat{u}(\xi) \in L^2(\mathbb{R}^n), \quad \xi \in \mathbb{R}^n
\]
(with $\hat{u}$ the Fourier transform of $u$).

We furnish $H^s(\mathbb{R}^n)$ with the norm
\[
\|u\|_s = \left( \int_{\mathbb{R}^n} (1 + |\xi|^2)^s |\hat{u}(\xi)|^2 d\xi \right)^{1/2}.
\]

For $s = m$ a non-negative integer, the space $H^s(\mathbb{R}^n)$ coincides with the usual space $H^m(\mathbb{R}^n)$.

• Thm: For $u \in H^1(\Omega)$, with $\Gamma = \partial \Omega$ of dimension $n - 1$ and piecewise of class $C^1$, we can define $u|_{\Gamma}$ (the trace of $u$ on $\Gamma$) as an element of $H^{1/2}(\Gamma)$.

Thm: For every $u_0 \in H^{1/2}(\Gamma)$, there is a $u \in H^1(\Omega)$ such that $u|_{\Gamma} = u_0$.

Note: For such set $\Gamma$, we can give a definition of $H^{1/2}(\Gamma)$ (with the aid of local maps defining $\Gamma$, see Lions-Magenes, Necas, Dautray-Lions, etc).

We also have another version of the Trace theorem:

Thm: Assume $\Omega$ is bounded and $\Gamma = \partial \Omega$ of class $C^1$. Then there exists a bounded linear operator
\[
T : H^1(\Omega) \rightarrow L^2(\Gamma)
\]
such that
(i) $Tu = u|_{\Gamma}$ if $u \in H^1(\Omega) \cap C(\overline{\Omega})$
(ii) $\|Tu\|_{L^2(\Gamma)} \leq C\|u\|_{H^1(\Omega)},$
for each $u \in H^1(\Omega)$, with constant $C$ depending only on $\Omega$.