

269C: HW#5 (final assignment) Due no later than Friday, June 14 (if you need additional time, please contact me by e-mail). After June 7, you can leave the final assignment in my mailbox, or with Babette Dalton in MS 7619 between 7-3pm, or you can slide it under the door of my office MS 7620-D.

[1] Consider the PDE (in distributional sense)

$$-\Delta u + k^2 u = f \quad \text{in } R^n.$$

Let $s \in R$. Show that, for all $f \in H^s(R^n)$, there exists a unique $u \in H^{s+2}(R^n)$, solution of the PDE, with $k \in R$, $k \neq 0$.

Hint: use the Fourier transform.

[2] Let $I = [0, h]$ and let $\pi v \in P_1(I)$ be the linear interpolant that agrees with $v \in C^2(I)$ at the end points of I . Using the technique of the proof of Thm. 4.1, prove estimates for $\|v - \pi v\|_{L^\infty(I)}$ and $\|v' - (\pi v)'\|_{L^\infty(I)}$, cf. (1.12) and (1.13).

[3] Using the general results from Chapter 4, estimate the error $\|u - u_h\|_{H^2}$ for Problem 1.5 and Example 2.4.

[4] Using polar coordinates (r, θ) , let $\Omega = \{(r, \theta) : 0 < r < 1, 0 < \theta < \omega\}$ be a pie-shaped domain of angle ω .

(i) Prove that the function $u(r, \theta) = r^\gamma \sin(\gamma\theta)$, $\gamma = \frac{\pi}{\omega}$, satisfies: $\Delta u = 0$ in Ω , $u = 0$ on the straight parts of the boundary of Ω .

(ii) Consider the cases $\omega > \pi$ and $\omega < \pi$ and study the condition $u \in H^2(\Omega)$ (see the discussion in Section 4.5).

[5] The following elliptic problem is approximated by the finite element method,

$$\begin{aligned} -\operatorname{div}(a(x)\nabla u(x)) &= f(x), \quad x \in \Omega \subset R^2, \\ u(x) &= 2, \quad x \in \partial\Omega_1, \\ \frac{\partial u(x)}{\partial x_1} + u(x) &= 0, \quad x \in \partial\Omega_2, \\ \frac{\partial u(x)}{\partial x_2} &= 0, \quad x \in \partial\Omega_3, \end{aligned}$$

where

$$\begin{aligned}\Omega &= \{(x_1, x_2) : 0 < x_1 < 1, 0 < x_2 < 1\}, \\ \Gamma_1 = \partial\Omega_1 &= \{(x_1, x_2) : x_1 = 0, 0 \leq x_2 \leq 1\}, \\ \Gamma_2 = \partial\Omega_2 &= \{(x_1, x_2) : x_1 = 1, 0 \leq x_2 \leq 1\}, \\ \Gamma_3 = \partial\Omega_3 &= \{(x_1, x_2) : 0 < x_1 < 1, x_2 = 0, 1\},\end{aligned}$$

and

$$0 < A \leq a(x) \leq B.$$

(a) Determine an appropriate weak formulation of the problem.

(b) Prove conditions on the corresponding linear and bilinear forms which are needed for existence and uniqueness and for the convergence of a finite element method (assume $f \in L^2(\Omega)$, $a \in L^\infty(\Omega)$).

(c) Describe briefly a finite element mesh, a FEM using P_1 elements, and a set of basis functions such that the linear system from the finite element approximation is sparse and of band structure.

[6] (a) Develop and describe the piecewise linear Galerkin finite element approximation of,

$$\begin{aligned}-\nabla \cdot a(x)\nabla u + b(x)u &= f(x), & x &= (x_1, x_2) \in \Omega, \\ u &= 2, & x &\in \partial\Omega_1, \\ \frac{\partial u}{\partial x_1} + \frac{\partial u}{\partial x_2} + u &= 2, & x &\in \partial\Omega_2,\end{aligned}$$

where $f \in L^2(\Omega)$,

$$\begin{aligned}\Omega &= \{x \mid x_1 > 0, x_2 > 0, x_1 + x_2 < 1\}, \\ \partial\Omega_1 &= \{x \mid x_1 = 0, 0 \leq x_2 \leq 1\} \cup \{x \mid x_2 = 0, 0 \leq x_1 \leq 1\}, \\ \partial\Omega_2 &= \{x \mid x_1 > 0, x_2 > 0, x_1 + x_2 = 1\}, \\ 0 < a &\leq a(x) \leq A, 0 < b \leq b(x) \leq B.\end{aligned}$$

(b) Justify the approximation by analyzing the appropriate bilinear and linear forms. Give a convergence estimate and quote the appropriate theorems for convergence.