HW #4, 269C Due during the week of May 27 - May 31st (note May 27 is a holiday).

[1] Recall the “coercive” inhomogeneous Neumann problem:

\[
\{(D) \quad \begin{cases}
-\Delta u + u = f \text{ in } \Omega, \\
\frac{\partial u}{\partial \vec{n}} = g \text{ on } \partial \Omega
\end{cases},
\]

We know that the corresponding weak variational problem (with \( V = \mathcal{H}^1(\Omega) \)) is: find \( u \in \mathcal{H}^1(\Omega) \) such that

\[
(V) \quad \int_{\Omega} (\nabla u \cdot \nabla v + uv) \, dx = \int_{\Omega} fv \, dx + \int_{\partial \Omega} gv \, ds
\]

for all \( v \in \mathcal{H}^1(\Omega) \). We also know that \((D) \Rightarrow (V)\).

Assume now that \( u, f, g \) are sufficiently smooth. Show that, if \( u \) is a smooth solution of \((V)\) (for example \( u \in C^2(\Omega) \)), then \( u \) satisfies \((D)\); in other words, \((V) \Rightarrow (D)\).

[2] Let \( K \) be a tetrahedron with vertices \( a^i, i = 1, \ldots, 4 \), and let \( a^{ij} \) denote the midpoint on the straight line \( a^i a^j \), \( i < j \). Show that a function \( v \in P_2(K) \) is uniquely determined by the degrees of freedom: \( v(a^i), v(a^{ij}), i, j = 1, \ldots, 4, i < j \). Show that the corresponding finite element space \( V_h \) satisfies \( V_h \subset C^0(\Omega) \).

[3] Let \( K \) be a triangle with vertices \( a^i, i = 1, 2, 3 \). Suppose that \( v \in P_r(K) \) and that \( v \) vanishes on the side \( a^2 a^3 \). Prove that \( v \) has the form

\[
v(x) = \lambda_1(x) w_{r-1}(x), \quad x \in K,
\]

where \( w_{r-1} \in P_{r-1}(K) \), and \( \lambda_1(x) \) is the affine local basis function corresponding to the node \( a^1 \).

[4] Let \( K \) be a triangle with vertices \( a^i, i = 1, 2, 3 \), and let \( a^{ij}, i < j \), denote the midpoints of the sides of \( K \). Let \( a^{123} \) denote the center of gravity of \( K \). Prove that \( v \in P_4(K) \) is uniquely determined by the following degrees of freedom

\[
\begin{align*}
v(a^i), \\
\frac{\partial v}{\partial x_j}(a^i), \quad i = 1, 2, 3, \quad j = 1, 2, \\
v(a^{ij}), \quad i, j = 1, 2, 3, \quad i < j, \\
v(a^{123}), \quad \frac{\partial v}{\partial x_j}(a^{123}), \quad j = 1, 2,
\end{align*}
\]

(typo in Exercise 3.8 in the textbook).

Also show that the functions in the corresponding finite element space \( V_h \) are continuous.