HW #3, 269C  Due on Monday, May 13

[1] Show that the Lax-Milgram theorem contains as particular case the Riesz representation theorem (in other words, assume Lax-Milgram and prove the Riesz representation theorem).

[2] Let $V$ be a Hilbert space, and $a : V \times V \to \mathbb{R}$ be a bilinear form. Show that: $a$ bounded is equivalent with $a$ continuous.

[3] Let $V$ be a Hilbert space and the (nonlinear) operator $A : V \to V$, satisfying

(i) there is $M \geq 0$ s.t. $\forall \ u, v \in V$, $\|Au - Av\| \leq M\| u - v\|$  
(ii) there is $\alpha > 0$ s.t. $\forall \ u, v \in V, \langle Au - Av, u - v \rangle \geq \alpha \| u - v \|^2$

Show that the nonlinear equation $Au = f$ has a unique solution (for $f \in V$), using the Banach fixed point theorem and the same technique for proving the Lax-Milgram theorem (introduce the function $g_\lambda$).

[4] Let $V$ be a complex Hilbert space, $a : V \times V \to \mathbb{C}$ a sesquilinear form, $L : V \to \mathbb{C}$ an anti-linear form. Assume that $a$ is Hermitian, thus $a(v, u) = \overline{a(u, v)}$, $\forall u, v \in V$, and that $a(v, v) \geq 0$. Consider the problems

(V) Find $u \in V$ s.t. $a(u, v) = L(v)$, $\forall v \in V$,

(M) Find $u \in V$ s.t. $J(u) = \inf_{v \in V} J(v)$,

with $J : V \to \mathbb{R}$ defined by $J(v) = \frac{1}{2}a(v, v) - \text{Re}L(v)$.

Show that $u \in V$ is solution of (V) iff $u \in V$ is solution of (M).

[5] Prove Poincaré inequality on $H^1_0(0, 1)$.

[6] Consider the problem with an inhomogeneous boundary condition,

$$\begin{cases} -\Delta u = f \text{ in } \Omega, \\ u = u_0 \text{ on } \Gamma = \partial\Omega, \end{cases}$$

where $f$ and $u_0$ are given. Show that this problem can be given the following equivalent variational formulations:

(V) Find $u \in V(u_0)$ such that $a(u, v) = (f, v)$, $\forall v \in H^1_0(\Omega)$,

(M) Find $u \in V(u_0)$ such that $F(u) \leq F(v)$, $\forall v \in V(u_0)$,

where $V(u_0) = \{ v \in H^1(\Omega) : v = u_0 \text{ on } \Gamma \}$.

Then formulate a finite element method and prove an error estimate (as in Thm. 1.1, page 24).

Recall: $H^1_0(\Omega) = \{ v \in L^2(\Omega), \nabla v \in L^2(\Omega)^n, v = 0 \text{ on } \partial\Omega \}$, where $n$ is the spatial dimension.
Useful notes

- Over the complex space $C$, with the above notations, we say that $a : V \times V \to C$ is a sesquilinear form if $a$ linear in the first argument (under the usual addition and scalar multiplication), $a$ antilinear in the second argument (usual addition but $a(u, \lambda v) = \bar{\lambda}a(u, v)$). Note that if $\text{Re}(v, v) \geq \alpha \|v\|^2$, together with $a$ and $L$ bounded, then the Lax-Milgram Lemma holds in the complex case with the same proof.

- Thm. Let $v \in H^1(a, b)$ and denote by $v' \in L^2(a, b)$ its first order distributional derivative. Then for almost every $x, y \in (a, b)$, we have

$$v(y) - v(x) = \int_x^y v'(t) dt.$$