HW #3, 269C   Due on Monday, May 13

[**1**] Show that the Lax-Milgram theorem contains as particular case the Riesz representation theorem (in other words, assume Lax-Milgram and prove the Riesz representation theorem).

[**2**] Let V be a Hilbert space, and $a : V \times V \to R$ be a bilinear form. Show that: a bounded is equivalent with a continuous.

[**3**] Let V be a Hilbert space and the (nonlinear) operator $A : V \to V$, satisfying

(i) there is $M \geq 0$ s.t. $\forall\, u, v \in V$, $\|Au - Av\| \leq M\|u - v\|$

(ii) there is $\alpha > 0$ s.t. $\forall\, u, v \in V$, $\langle Au - Av, u - v\rangle \geq \alpha\|u - v\|^2$

Show that the nonlinear equation $Au = f$ has a unique solution (for $f \in V$), using the Banach fixed point theorem and the same technique for proving the Lax-Milgram theorem (introduce the function g_λ).

[**4**] Let V be a complex Hilbert space, $a : V \times V \to C$ a sesquilinear form, $L : V \to C$ an anti-linear form. Assume that a is Hermitian, thus $a(v, u) = \overline{a(u, v)}$, $\forall u, v \in V$, and that $a(v, v) \geq 0$. Consider the problems

$$\text{(V) Find } u \in V \text{ s.t. } a(u, v) = L(v),\ \forall v \in V,$$

$$\text{(M) Find } u \in V \text{ s.t. } J(u) = \inf_{v \in V} J(v),$$

with $J : V \to R$ defined by $J(v) = \frac{1}{2}a(v, v) - \mathrm{Re}L(v)$.

Show that $u \in V$ is solution of (V) iff $u \in V$ is solution of (M).

[**5**] Prove Poincaré inequality on $H_0^1(0, 1)$.

[**6**] Consider the problem with an inhomogeneous boundary condition,

$$\begin{cases} -\triangle u = f \text{ in } \Omega, \\ u = u_0 \text{ on } \Gamma = \partial\Omega, \end{cases}$$

where f and u_0 are given. Show that this problem can be given the following equivalent variational formulations:

(V) Find $u \in V(u_0)$ such that $a(u, v) = (f, v)$, $\forall v \in H_0^1(\Omega)$,

(M) Find $u \in V(u_0)$ such that $F(u) \leq F(v)$, $\forall v \in V(u_0)$,

where $V(u_0) = \{v \in H^1(\Omega) :\ v = u_0 \text{ on } \Gamma\}$.

Then formulate a finite element method and prove an error estimate (as in Thm. 1.1, page 24).

Recall: $H_0^1(\Omega) = \{v \in L^2(\Omega),\ \nabla v \in L^2(\Omega)^n,\ v = 0 \text{ on } \partial\Omega\}$, where n is the spatial dimension.

Useful notes

• Over the complex space C, with the above notations, we say that $a : V \times V \to C$ is a is sesquilinear form if a linear in the first argument (under the usual addition and scalar multiplication), a antilinear in the second argument (usual addition but $a(u, \lambda v) = \bar{\lambda} a(u, v)$). Note that if $\mathrm{Re} a(v, v) \geq \alpha \|v\|^2$, together with a and L bounded, then the Lax-Milgarm Lemma holds in the complex case with the same proof.

• **Thm.** Let $v \in H^1(a, b)$ and denote by $v' \in L^2(a, b)$ its first order distributional derivative. Then for almost every $x, y \in (a, b)$, we have

$$v(y) - v(x) = \int_x^y v'(t) dt.$$