[1] Find the linear basis functions for the triangle $K$ with vertices at $(0,0)$, $(h,0)$ and $(0,h)$. Show that the corresponding element stiffness matrix is given by

$$
\begin{bmatrix}
1 & -\frac{1}{2} & -\frac{1}{2} \\
-\frac{1}{2} & \frac{1}{2} & 0 \\
-\frac{1}{2} & 0 & \frac{1}{2}
\end{bmatrix}
$$

Using this result show that the linear system (1.25) of Example 1.1. has the stated form (pages 30-31) (there is a typo in the textbook regarding the linear system).

[2] (a) Give a weak variational formulation of the problem

$$
d^4u/dx^4 = f \quad \text{for} \quad 0 < x < 1, $$

$$
u(0) = u''(0) = u'(1) = u''(1) = 0,$$

and show that the assumptions of the Lax-Milgram Lemma are satisfied. Which boundary conditions are essential and which are natural ?

(b) Solve the same problem with the following alternative boundary conditions:

$$
u(0) = -u''(0) + \gamma u'(0) = 0, \quad u(1) = u''(1) + \gamma u'(1) = 0,$$

where $\gamma$ is a positive constant.


$$
-\Delta u = f \quad \text{in} \quad \Omega, \quad \gamma u + \frac{\partial u}{\partial n} = g \quad \text{on} \quad \Gamma,
$$

where $\gamma$ is a constant. When are the assumptions of the Lax-Milgram Lemma satisfied ?

[4] Consider the Neumann problem

$$
-\Delta u = f \quad \text{in} \quad \Omega,$$

$$
\frac{\partial u}{\partial n} = g \quad \text{on} \quad \Gamma = \partial \Omega,$$

$$
\int_{\Omega} u(x)dx = 0.
$$

where $f : \Omega \to R$ and $g : \partial \Omega \to R$ satisfy the compatibility condition

$$
\int_{\Omega} f(x)dx + \int_{\partial \Omega} g(x)d\sigma(x) = 0.
$$

(a) Why condition “$\int_{\Omega} u(x)dx = 0$” was added here ? Why do we need the compatibility condition ?

(b) Give a weak variational formulation of the problem, and prove that the conditions of the Lax-Milgram Lemma are satisfied, under the necessary assumptions on $f$ and $g$ that you would specify.