HW#1, 269C, Due on Monday, April 15

[1] Show that if w is continuous on $[0,1]$, and

$$\int_0^1 wv dx = 0, \text{ for all } v \in V,$$

with $V = \{v : [0,1] \to R,\ \textit{continuous},\ v(0) = v(1) = 0,\ v' \ \textit{piecewise} - \textit{continuous and bounded}\}$, then $w(x) = 0$ for $x \in [0,1]$.

[2] Construct a finite-dimensional subspace V_h of V (from problem [1]) consisting of functions which are quadratic on each subinterval I_j of a partition of $I = (0,1)$. How can one choose the parameters to describe such functions ? Find the corresponding basis functions. Then formulate a finite element method for (D) using the space V_h and write down the corresponding linear system of equations in the case of a uniform partition. Recall that (D) is

$$-u'' = f \text{ in } (0,1), \quad u(0) = u(1) = 0.$$

[3] Consider the BVP

$$\frac{d^4u}{dx^4} = f, \quad 0 < x < 1, \quad u(0) = u'(0) = u(1) = u'(1) = 0.$$

(a) Show that this problem can be given the following variational formulation: Find $u \in W$ such that

$$(u'', v'') = (f, v), \quad \forall v \in W,$$

where $W = \{v : v$ and v' are continuous on $[0,1]$, v'' is piecewise-continuous and $v(0) = v'(0) = v(1) = v'(1) = 0\}$.

(b) For $I = [a,b]$ an interval, define

$$P_3(I) = \{v : \ v \text{ is a polynomial of degree } \leq 3 \text{ on } I\},$$

i.e. v has the form $v(x) = a_3x^3 + a_2x^2 + a_1x + a_0$ for $x \in I$, and $a_i \in R$.

Show that $v \in P_3(I)$ is uniquely determined by the values $v(a)$, $v'(a)$, $v(b)$, $v'(b)$. Find the corresponding basis functions (the basis function corresponding to the value $v(a)$ is the cubic polynomial v such that $v(a) = 1$, $v'(a) = 0$, $v(b) = 0$, $v'(b) = 0$, etc.)

(c) Starting from (b) construct a finite-dimensional subspace W_h of W consisting of piecewise-cubic functions. Specify suitable parameters to describe the functions in W_h and determine the corresponding basis functions.

(d) Formulate a FEM for the problem based on the space W_h. Find the corresponding linear system of equations in the case of a uniform partition.