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Practice problems

[1] Write the differential equation

$$
\begin{aligned}
-\triangle u + u = f(x,y), &\quad (x,y) \in \Omega \\
u = 1 &\quad (x,y) \in \partial\Omega_1 \\
\frac{\partial u}{\partial n} + u = x &\quad (x,y) \in \partial\Omega_2,
\end{aligned}
$$

where
$\Omega = \{(x,y) |\ x^2 + y^2 < 1\}$,
$\partial\Omega_1 = \{(x,y) |\ x^2 + y^2 = 1,\ x \le 0\}$,
$\partial\Omega_2 = \{(x,y) |\ x^2 + y^2 = 1,\ x > 0\}$,
in a weak variational form and describe a piecewise-linear Galerkin finite element approximation for the problem. Analyze the assumptions of the Lax-Milgram theorem.

[2] (a) Develop and describe the piecewise linear Galerkin finite element approximation of,

$$
\begin{aligned}
-\nabla \cdot a(x)\nabla u + b(x)u = f(x), &\quad x = (x_1, x_2) \in \Omega, \\
u = 2, &\quad x \in \partial\Omega_1, \\
\frac{\partial u}{\partial x_1} + \frac{\partial u}{\partial x_2} + u = 2, &\quad x \in \partial\Omega_2,
\end{aligned}
$$

where
$\Omega = \{x |\ x_1 > 0,\ x_2 > 0,\ x_1 + x_2 < 1\}$,
$\partial\Omega_1 = \{x |\ x_1 = 0,\ 0 \le x_2 \le 1\} \cup \{x |\ x_2 = 0,\ 0 \le x_1 \le 1\}$,
$\partial\Omega_2 = \{x |\ x_1 > 0,\ x_2 > 0,\ x_1 + x_2 = 1\}$,
$0 < a \le a(x) \le A$, $0 < b \le b(x) \le B$.
(b) Justify the approximation by analyzing the appropriate bilinear and linear forms. Give a convergence estimate and quote the appropriate theorems for convergence.

[3] Consider the elliptic boundary value problem

$$
-\frac{d}{dx}\Big[(1+x)\frac{du}{dx}\Big] + \frac{u}{1+x} = \frac{2}{1+x}, \quad 0 < x < 1,
$$

$$u(0) = 0, \quad u(1) = 1.$$

(a) Give a weak formulation for the problem.
(b) Verify the assumptions of the Lax-Milgram lemma.
(c) Setup a finite element approximation for this problem.
Note an alternative approach: let $w(x) = u(x) - x$, then $w(0) = w(1) = 0$.

[4] Develop and describe the piecewise-linear Galerkin finite element approximation of

$$\begin{aligned} -\triangle u + u &= f(x,y), \quad (x,y) \in T, \\ u &= g_1(x), \quad (x,y) \in T_1, \\ u &= g_2(y), \quad (x,y) \in T_2, \\ \frac{\partial u}{\partial n} &= h(x,y), \quad (x,y) \in T_3, \end{aligned}$$

where

$$\begin{aligned} T &= \{(x,y) |\ x > 0,\ y > 0,\ x + y < 1\} \\ T_1 &= \{(x,y) |\ y = 0,\ 0 < x < 1\} \\ T_2 &= \{(x,y) |\ x = 0,\ 0 < y < 1\} \\ T_3 &= \{(x,y) |\ x > 0,\ y > 0,\ x + y = 1\}. \end{aligned}$$

Justify your approximation by analyzing the appropriate bilinear and linear forms. Give a weak formulation of the problem. Give a convergence estimate and quote the appropriate theorems for convergence.

[5] Develop and describe the piecewise linear Galerkin finite element approximation of

$$\begin{cases} -\triangle u + b(x)u = f(x), & x = (x_1, x_2) \in \Omega \\ u = 2, & x \in \partial\Omega_1 \\ \frac{\partial u}{\partial x_1} + \frac{\partial u}{\partial x_2} + u = 2, & x \in \partial\Omega_2, \end{cases}$$

where

$$\begin{aligned} \Omega &= \{x | x_1 > 0,\ x_2 > 0,\ x_1 + x_2 < 1\} \\ \partial\Omega_1 &= \{x | x_1 = 0,\ 0 \le x_2 \le 1\} \cup \{x | x_2 = 0,\ 0 \le x_1 \le 1\} \\ \partial\Omega_2 &= \{x | x_1 > 0,\ x_2 > 0,\ x_1 + x_2 = 1\} \end{aligned}$$

and

$$0 < b \leq b(x) \leq B.$$

Justify your approximation by analyzing the appropriate bilinear and linear forms. Give a weak formulation of the problem. Give a convergence estimate and quote the appropriate theorems for convergence

[6] Consider the following problem in a domain $\Omega \subset R^2$, with $\Gamma = \partial\Omega$:

$$\begin{aligned} -\triangle u + \beta_1 \frac{\partial u}{\partial x_1} + \beta_2 \frac{\partial u}{\partial x_2} + u = f \text{ in } \Omega, \\ u = 0 \text{ on } \Gamma, \end{aligned}$$

where the β_i are constants.

(a) Choose an appropriate space of test functions V, and give a weak formulation of the problem.

(b) For any $v \in V$, show that

$$\int_\Omega \Big(\beta_1 \frac{\partial v}{\partial x_1} v + \beta_2 \frac{\partial v}{\partial x_2} v\Big) dx = 0.$$

(c) By analyzing the linear and bilinear forms, show that the weak formulation has a unique solution.

(d) Set up a convergent finite element approximation and discuss the linear system thus obtained.

Additional practice problems

(some problems were given at past numerical analysis qualifying exams)

[**1**] Let $n \geq 2$ be an integer and $\Omega \subset R^n$ a bounded domain with Lipschitz boundary $\Gamma = \partial\Omega$. Let $a_{ij} \in L^\infty(\Omega)$ for all $i, j = 1, ..., n$, and assume that there exists a constant $\lambda > 0$ such that

$$\sum_{i,j=1}^{n} a_{ij}(x)\xi_i\xi_j \geq \lambda|\xi|^2, \text{ for all } x \in \Omega,\ \xi \in R^n.$$

Let $b \in L^\infty(\Omega)$ with $b \geq 0$ a.e. in Ω and $f \in L^2(\Omega)$. Moreover, let $\Gamma_0 \subset \Gamma$ and $\Gamma_1 = \Gamma \setminus \Gamma_0$, be both dS-measurable subsets of Γ with positive dS-measures.

Consider the problem

$$(P) \qquad \begin{array}{c} -\sum_{i,j=1}^{n} \partial_{x_j}(a_{ij}\partial_{x_i}u) + bu = f \text{ in } \Omega, \\ u = 0 \text{ on } \Gamma_0, \\ \sum_{i,j=1}^{n} a_{ij}(\partial x_i u)n_j = g \text{ on } \Gamma_1, \end{array}$$

where $\vec{n} = (n_1, ..., n_n)$ is the unit exterior normal along the boundary $\partial\Omega$.

(a) Give a weak variational formulation (V) of the problem, and show that this weak problem has a unique solution.

(b) If in addition $a_{ij} \in W^{1,\infty}(\Omega)$ $(i, j = 1, ..., n)$ and $u \in C^2(\overline{\Omega})$, show that (V) implies (P).

(c) Setup a convergent finite element formulation of the problem using P_1 elements (show the main properties of the linear system, show an abstract stability estimate, and give a rate of convergence).

[**2**] The following elliptic problem is approximated by the finite element method,

$$\begin{aligned} -\nabla \cdot \Big(a(x)\nabla u(x)\Big) &= f(x),\ x \in \Omega \subset R^2, \\ u(x) &= u_0,\ x \in \Gamma_1, \\ \frac{\partial u(x)}{\partial x_1} + u(x) &= 0,\ x \in \Gamma_2, \\ \frac{\partial u(x)}{\partial x_2} &= 0,\ x \in \Gamma_3, \end{aligned}$$

where

$$\Omega = \{(x_1, x_2) :\ 0 < x_1 < 1,\ 0 < x_2 < 1\},$$

$$\begin{aligned}
\Gamma_1 &= \{(x_1, x_2): \ x_1 = 0, \ 0 \le x_2 \le 1\}, \\
\Gamma_2 &= \{(x_1, x_2): \ x_1 = 1, \ 0 \le x_2 \le 1\}, \\
\Gamma_3 &= \{(x_1, x_2): \ 0 < x_1 < 1, \ x_2 = 0, \ 1\},
\end{aligned}$$

$$0 < A \le a(x) \le B, \quad a.e. \text{ in } \Omega, \quad f \in L^2(\Omega),$$

and $u_0|_{\Gamma_1}$ is the trace of a function $u_0 \in H^1(\Omega)$.

(a) Determine an appropriate weak variational formulation of the problem.

(b) Prove conditions on the corresponding linear and bilinear forms which are needed for existence and uniqueness of the solution.

(c) Describe a FEM using P_1 elements, and a set of basis functions such that the linear system from the finite element approximation is sparse and of band structure. Discuss the linear system thus obtained, and give a rate of convergence.

[**3**] Let A be a 2×2 symmetric matrix (can have space-dependent entries). Let $\nabla V = \begin{pmatrix} v_x \\ v_y \end{pmatrix}$, Ω be the unit square.

(a) Give conditions on A and the space of functions S, so that the problem

$$\min_{v \in S} \Big\{ \frac{1}{2} \int_\Omega (\nabla V)^T A (\nabla V) dx dy - \int_\Omega f v dx dy \Big\},$$

has a minimum for $f \in L^2(\Omega)$, where $v = 0$ on the boundary of Ω (note, T denotes transpose).

(b) For those A, setup a finite element method that converges and obtain the rate.

(c) Justify your statements.

[**4**] Consider the differential equation

$$u_{xx} + 2u_{yy} - 3u_x - 4u = f(x, y), \ (x, y) \in \Omega,$$

$$\frac{\partial u}{\partial \vec{n}} = g(x, y), \ (x, y) \in \partial\Omega,$$

where Ω is the unit square.

(a) Derive a Galerkin finite element approximation of the problem.

(b) Obtain the conditions on the appropriate bilinear and linear forms that guarantee convergence of the finite element method.

(c) Determine the diagonal elements in the element stiffness matrix for $P_1(K)$ elements. The triangle K has the vertices $(0,0)$, $(0,h)$ and $(h,0)$.

[5] Consider the Neumann problem

$$(A) \quad -(u_{xx}+u_{yy}) = f(x,y),\ -1<x<1,\ -1<y<1,$$

with

$$(B) \quad \frac{\partial u}{\partial \vec{n}} = g$$

($\vec{n}$ is the outwards unit normal) and the condition

$$(C) \quad \int_{|x|<1,|y|<1} u(x,y)dxdy = 0.$$

(a) Why do we need condition (C) ?

Now replace (A) by

$$(A') \quad u-(u_{xx}+u_{yy}) = f$$

and keep condition (B).

(b) Do we still need condition (C) ? Why or why not ?

(c) Set up a finite element method that converges for the problem (A'), (B). Justify your answers.

[6] Consider the following partial differential equations

$$-\frac{\partial}{\partial x}\Big(a(x,y)\frac{\partial u}{\partial x}\Big) - \frac{\partial}{\partial y}\Big(b(x,y)\frac{\partial u}{\partial y}\Big) + c(x,y)u = f(x,y),\ (x,y)\in\Omega$$

$$u = 1,\ (x,y)\in\partial\Omega_1$$

$$\frac{\partial u}{\partial y} = 0,\ (x,y)\in\partial\Omega_2$$

where $\Omega = [0,1]^2$, $\partial\Omega_1 = \{(x,y),\ |x|=1,\ |y|\le 1\}$, $\partial\Omega_2 = \{(x,y),\ |y|=1,\ |x|<1\}$.

(a) Set up a finite element method based on a weak form of the problem above.

(b) Give conditions on a, b and c such that the method will converge. Give the convergence estimate and motivate your answers.

[7] Consider the evolution problem

$$\frac{\partial u}{\partial t} = \nabla \cdot \Big(a(x) \nabla u \Big), \ x \in \partial\Omega \subset R^2, \ t > 0, \ a \geq a_0 > 0$$

$$\frac{\partial u}{\partial \vec{n}} + bu = f(x), \ x \in \partial\Omega, \ t > 0$$

$$u(x, 0) = u_0(x), \ x \in \Omega.$$

(a) Give a weak formulation of the problem.

(b) Describe how to use the Galerkin method together with Crank-Nicolson discretization in time to obtaina numerical method based on piecewise-linear elements.

(c) Show that the matrices that need to be inverted at each time step are nonsingular for $b = 0$.

[8] (a) Derive a weak variational formulation of the convection-diffusion prolem,

$$-\triangle u + \vec{a} \cdot \nabla u + bu = f(x, y) \ 0 < x < 1, \ 0 < y < 1$$

$$u = c(x, y), \ x = 0, 1, \ 0 \leq y \leq 1$$

$$\frac{\partial u}{\partial \vec{n}} = d(x, y) \ 0 < x < 1, \ y = 0, 1$$

where $\vec{a}$, b, c, d, and f are smooth functions.

(b) Under what assumptions on the coefficients $\vec{a}$, b, we obtain a convergent finite element approximation ?

[9] Let Ω, Ω_i, $i = 1, 2$ be bounded Lipschitz domains in R^2, such that $\Omega_i \subset \Omega$ $(i = 1, 2)$, $\overline{\Omega} = \overline{\Omega}_1 \cup \overline{\Omega}_2$, and each $\Gamma_i := \partial\Omega_i \cap \partial\Omega$ has a positive dS-measure. Denote $S = \partial\Omega_1 \cap \partial\Omega_2$. Consider the interface boundary value problem

$$-\nabla \cdot (a \nabla u) = f \text{ in } \Omega,$$

$$u = 0 \text{ on } \partial\Omega,$$

$$[u] = [a \partial_\nu u] = 0 \text{ on } S,$$

where

$$a(x) = \begin{cases} a_1 \text{ if } x \in \Omega_1 \\ a_2 \text{ if } x \in \Omega_2 \end{cases},$$

and a_1, a_2 are two distinct, positive, real numbers, $f \in L^2(\Omega)$, ν is the unit exterior normal of $\partial\Omega_2$, and $[\cdot]$ denotes the jump across the interface S.

(a) Find the weak formulation of the boundary value problem.

(b) Prove that the problem in weak formulation has a unique solution.

(c) Prove that the weak solution, if it is smooth enough, solves the boundary value problem (for instance, assume u weak solution and $u \in C^2(\overline{\Omega}_i)$, $i = 1, 2$).

[10] Consider the Newmann problem

$$-\triangle u = 0 \text{ in } \Omega,$$

$$\frac{\partial u}{\partial \vec{n}} = g \text{ on } \partial\Omega,$$

where Ω is sufficiently smooth and $g \in L^2(\partial\Omega)$.

(a) Give a weak variational formulation of the problem.

(b) Give a condition on g necessary to guarantee the existence of a solution to this problem.

(c) Give a condition on u necessary to guarantee the existence and uniqueness of a solution to this problem.