Evolution problems of first order in time

1.1 Function spaces
We are given a pair of real, separable Hilbert spaces \( V, H \); we denote by \((,\)) the scalar product, \(||\cdot||\) the norm in \( V \)
\langle,\rangle the scalar product, \(||\cdot|||\) the norm in \( H \).
We suppose \( V \) is dense in \( H \) and we identify \( H \) with its dual \( H' \). We also denote the duality between \( V' \) and \( V \) by \((,\).

1.2 The bilinear form \( a(t; u, v), t \in [0, T] \)
For each \( t \in [0, T] \), we are given a continuous bilinear form over \( V \times V \) and we make the hypothesis:

(3.3) For every \( u, v \in V \), the function \( t \to a(t; u, v) \) is measurable and there is a constant \( M = M(T) > 0 \) (independent of \( t \in [0, T], u, v \)) such that

\[ |a(t; u, v)| \leq M \|u\|\|v\| \]

for all \( u, v \in V \).

Def. Let \( a, b \in \mathbb{R} \). Then

\[ W(V) = W(a, b; V, V') = \{u; u \in L^2(a, b; V), u' \in L^2(a, b; V')\} \]

Proposition. This is a Hilbert space equipped with the norm

\[ \|u\|_W = (\|u\|_{L^2(a, b; V)}^2 + \|u'\|_{L^2(a, b; V')}^2)^{1/2} = \left( \int_a^b (\|u(t)\|_V^2 + \|u'(t)\|_{V'}^2) dt \right)^{1/2} \]

We also assume (3.25) \( a(t; u, u) \geq \alpha \|u\|_V^2 \), for any \( t \in [0, T], u \in V \), and \( u_0 \in H, f \in L^2(V') \).

Evolution Problem (P) Find \( u \) satisfying \( u \in W(V), \)

\[ \frac{d}{dt}(u(\cdot), v) + a(\cdot; u(\cdot), v) = (f(\cdot), v) \]
in the sense of distributions \( D'(\mathbb{R}) \) for all \( v \in V, u(0) = u_0 \).

Remark. We have

\[ \frac{d}{dt}(u(\cdot), v) = \left( \frac{d}{dt}u(\cdot), v \right), \]

for any \( v \in V \).

Theorem 1. Then the solution of problem (P), if it exists, is unique.

Proof. Let \( u_1, u_2 \) be two distinct solutions of (P), then \( w = u_1 - u_2 \) satisfies \( w \in W(V) \) and

\[ \left( \frac{dw}{dt}(\cdot), v \right) + a(\cdot; w(\cdot), v) = 0, \]

for any \( v \in V \), with \( w(0) = 0 \). Then by replacing \( v \) by \( w(t) \) and integrating from 0 to \( t \):

\[
\frac{1}{2} |w(t)|^2 + \int_0^t a(s; w(s), w(s)) \, ds = 0.
\]

Since \( a(\cdot; u, v) \) is \( V - elliptic \), we have then

\[
\frac{1}{2} |w(t)|^2 < 0 \implies w(t) = 0 \text{ for all } t \in [0, T].
\]

**Theorem 2.** There exists a solution \( u \) to problem (P), and \( u \in W(0, T; V, V') \).

**Examples**

1. Let \( \Omega \) be an open and bounded subset of \( \mathbb{R}^n \), with boundary \( \Gamma \), \( T \) finite, \( V = H^1_0(\Omega), H = L^2(\Omega), V' = H^{-1}(\Omega) \). Let \( \Omega_T = \Omega \times [0, T[, \Gamma_T = \Gamma \times [0, T[ \).

   The following problem
   
   \[ \frac{\partial u}{\partial t} - \Delta u = f, \quad u_{|\Gamma_T} = 0, \quad u(\cdot, 0) = u_0 \text{ in } \Omega \]

   has a unique solution using the bilinear form
   
   \[ a(t; u, v) = (\nabla u, \nabla v), \quad \text{for } t \in [0, T], \]

   assuming \( f \in L^2(0, T; H^{-1}(\Omega)) \), \( u_0 \in L^2(\Omega) \).

2. If we consider \( V = H^1(\Omega) \) instead, \( H = L^2(\Omega) \), and if \( a \) satisfies

   \[ a(t; u, u) + \lambda |u|^2 \geq \alpha \|u\|^2, \quad t \in [0, T], \quad u \in V, \]

   then using \( a \) as in Example 1, we formally obtain that the Cauchy-Neumann problem has a unique solution:

   \[ \frac{\partial u}{\partial t} - \Delta u = f, \quad \frac{\partial u}{\partial n}_{|\Gamma_T} = f_1, \quad u(\cdot, 0) = u_0 \text{ in } \Omega. \]

3. If \( f \) is such that, for any \( v \in H^1(\Omega) \):

   \[ (f(t), v) = \int_{\Omega} f_0 v dx + \int_{\Gamma} f_1 v d\Gamma, \]

   where \( f_0 \in L^2(0, T; L^2(\Omega)) \) and \( f_1 \in L^2(0, T; H^{-1/2}(\Gamma)) \), then \( f \in L^2(0, T; V') \) and the corresponding problem is

   \[ \frac{\partial u}{\partial t} - \Delta u = f, \quad \frac{\partial u}{\partial n}_{|\Gamma_T} = f_1, \quad u(\cdot, 0) = u_0 \text{ in } \Omega. \]

4. Mixed Dirichlet-Neumann BC can be considered.