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Useful results 2
Additional results and remarks

(see "Finite Elements, Mathematical Aspects", Vol. IV, J. Tinsley Oden and Graham F. Carey, Prentice Hall, 1983)

Thm. *The Generalized Lax-Milgram Theorem:* Let H and G be real Hilbert spaces and let $B(\cdot,\cdot)$ denote a bilinear form on $H \times G$ which has the following properties:

(i) $B(\cdot,\cdot)$ is continuous; that is, there exists a constant $M > 0$ such that

$$|B(u,v)| \leq M\|u\|_H\|v\|_G, \text{ for any } u \in H,\ v \in G.$$

(ii) $B(\cdot,\cdot)$ is coercive in the sense that there exists a constant α such that

$$\inf_{u\in H,\ \|u\|_H=1}\ \sup_{v\in G,\ \|v\|_G\leq 1} |B(u,v)| \geq \alpha > 0.$$

(iii) For every $v \neq 0$ in G,

$$\sup_{u\in H} |B(u,v)| > 0.$$

Then there exists a unique $u^* \in H$ such that

$$B(u^*,v) = F(v), \text{ for all } v \in G,$$

wherein $F \in G'$. Moreover,

$$\|u^*\|_H \leq \frac{1}{\alpha}\|F\|_{G'}.$$

If $H = G$, then (ii) and (iii) can be replaced by the simpler condition:
(iv) There exists an $\alpha > 0$ such that

$$B(u,u) \geq \alpha\|u\|_H^2, \text{ for all } u \in H.$$

(in this case, we say that B is $H - coercive$).

Here G' denotes the dual of G, the space of continuous linear forms F on H.

Mixed Methods

Notations and conventions:

• H, Q are real Hilbert spaces with norms $\|\cdot\|_H$ and $\|\cdot\|_Q$ respectively.

• $a : H \times H$ is a symmetric, continuous bilinear form so that $M > 0$ exists such that

$$|a(u,v)| \leq M\|u\|_H\|v\|_H, \text{ for all } u,v \in H.$$

• $b : H \times Q$ is a continuous bilinear form so that $m > 0$ exists such that

$$|b(u,q)| \leq m\|u\|_H\|q\|_Q, \text{ for all } u \in H,\ q \in Q.$$

• $f : H \to R$ is a continuous linear form on H.

• $g : Q \to R$ is a continuous linear form on Q.

We consider the weak variational formulation

$$a(u,v) + b(v,p) = f(v), \text{ for all } v \in H \quad (1)$$

$$b(u,q) = g(q) \text{ for all } q \in Q. \quad (2)$$

This corresponds to the abstract problem (in distributional sense)

$$Au + B^*p = f \text{ in } H',$$

$$Bu = g \text{ in } Q'.$$

We also define:

$$kerB = \{v \in H | b(v,q) = 0 \text{ for all } q \in Q\},$$

$$kerB^* = \{q \in Q | b(v,q) = 0 \text{ for all } v \in H\},$$

and the quotient space

$$Z = Q/kerB^*.$$

The quotient space Z consists of equivalence classes $[p]$ of the form

$$[p] = \{q \in Q | p - q \in kerB^*\}.$$

We also assume that $g \in Rg(B)$.

Note that $Bu = g$ is a constraint, corresponding to an unknown Lagrange multiplier p in (1).

Problem (1)-(2) can be substituted by the following equivalent problem: introduce a bilinear form $B(\cdot,\cdot)$ defined on the product space $H \times Q$,

$B : (H \times Q) \times (H \times Q) \to R$, by

$$B((u,p),(v,q)) = a(u,v) + b(v,p) + b(u,q)$$

and the linear form $F : H \times Q \to R$,

$$F((v,q)) = f(v) + g(q).$$

Then (1)-(2) can be written: find $(u,p) \in H \times Q$ s.t.

$$B((u,p),(v,q)) = F((v,q)), \text{ for all } (v,q) \in H \times Q. \tag{3}$$

Also impose conditions: there is $\alpha_0 > 0$ s.t.

$$\alpha_0 \|u_0\|_H \leq \sup_{v_0 \in kerB - \{0\}} \frac{|a(u_0,v_0)|}{\|v_0\|_H}, \text{ for all } u_0 \in kerB, \tag{4}$$

and there is $\beta >$ s.t.

$$\beta \inf_{p_0 \in kerB^*} \|p + p_0\|_Q = \beta \|[p]\|_Z \leq \sup_{v \in H - \{0\}} \frac{|b(v,p)|}{\|v\|_H}, \text{ for all } p \in Q. \tag{5}$$

Conditions (4) and (5) are equivalent with: there is $\alpha > 0$ s.t. for all $(u,p) \in H \times Q$,

$$\alpha(\|u\|_H + \|[p]\|_Z) \leq \sup_{(v,q) \in H \times Q, (v,q) \neq (0,0)} \frac{|a(u,v) + b(v,p) + b(u,q)|}{\|v\|_H + \|q\|_Q}.$$

Thm. Let conditions (4) and (5) hold for the continuous bilinear forms $a(\cdot,\cdot)$ and $b(\cdot,\cdot)$ defined in the beginning. Then there exists a unique solution $(u,[p]) \in H \times Z$, $Z = Q/kerB^*$, of problem (3) with $g \in Rg(B)$. The Lagrange multiplier p is then unique up to an arbitrary element of $kerB^*$.

- Apply the above mixed method to the stationary Stokes equations.