269C, Vese
Useful results

Notations:
For $u \in H^m(\Omega)$, let

$$\|u\|_{H^m(\Omega)} = \Big(\sum_{|\alpha| \leq m} \int_\Omega |D^\alpha u(x)|^2 dx \Big)^{1/2} = \Big(\sum_{|\alpha| \leq m} \|D^\alpha u\|^2_{L^2(\Omega)} \Big)^{1/2},$$

$$|u|_{H^m(\Omega)} = \Big(\sum_{|\alpha| = m} \int_\Omega |D^\alpha u(x)|^2 dx \Big)^{1/2} = \Big(\sum_{|\alpha| = m} \|D^\alpha u\|^2_{L^2(\Omega)} \Big)^{1/2}.$$

Thm. (Poincaré's Inequality for $H^1_0(\Omega)$)
Let Ω be an open and bounded set in R^n. Then there is a positive constant $C = C(\Omega)$ such that, for all $u \in H^1_0(\Omega)$, we have Poincaré inequality:

$$\|u\|^2_{L^2(\Omega)} \leq C \|\nabla u\|^2_{L^2(\Omega)}.$$

Corollary: Let $m > 0$ be a positive integer, and let Ω be an open and bounded set in R^n. Then for $u \in H^m_0(\Omega)$, we have

$$\|u\|^2_{L^2(\Omega)} \leq C_m C^m \sum_{|\alpha| = m} \|D^\alpha u\|^2_{L^2(\Omega)},$$

C_m =constant, and C is the constant from the previous theorem.

Corollary: (same assumptions on Ω). $|u|_{H^m(\Omega)}$ is a norm on $H^m_0(\Omega)$, equivalent to the norm $\|u\|_{H^m(\Omega)}$.

Thm. Let Ω be a bounded connected open set in R^n, with sufficiently regular boundary. Then we have for $u \in H^1(\Omega)$, such that $\int_\Omega u(x)dx = 0$,

$$\|u\|^2_{L^2(\Omega)} \leq P(\Omega) \|\nabla u\|^2_{L^2(\Omega)}.$$

More generally, we have for $u \in H^1(\Omega)$

$$\|u\|^2_{L^2(\Omega)} \leq P(\Omega) \|\nabla u\|^2_{L^2(\Omega)} + \frac{1}{|\Omega|} \Big| \int_\Omega u(x)dx \Big|^2.$$

Corollary: $|u|_{H^1(\Omega)} = \|\nabla u\|_{L^2(\Omega)}$ is a norm equivalent with the norm $\|u\|_{H^1(\Omega)}$ on the sub-space V_0 (closed in $H^1(\Omega)$) defined by:

$$V_0 = \{u \in H^1(\Omega) : \int_\Omega u(x)dx = 0\}.$$

Corollary: Let Ω be a bounded connected open set in R^n, with sufficiently regular boundary Γ. Suppose $\Gamma = \Gamma_1 \cup \Gamma_2$ with length (area) of $\Gamma_2 > 0$. Let

$$V_{\Gamma_2} = \{u \in H^1(\Omega) : \ u|_{\Gamma_2} = 0\}.$$

Then V_{Γ_2} is a closed sub-space of $H^1(\Omega)$ and $|u|_{H^1(\Omega)} = \|\nabla u\|_{L^2(\Omega)}$ is a norm equivalent with the norm $\|u\|_{H^1(\Omega)}$ on the sub-space V_{Γ_2}.

Remark:

(i) Suppose that Ω is a bounded connected open set in R^n which is "very regular" ($\Gamma = \partial\Omega$ is a $n-1$ dimensional manifold of class C^∞ and Ω locally on one side of Γ). For $u \in H^1(\Omega)$, let

$$\|u\|^2_{H^1(\Omega),\Gamma} = \|\nabla u\|^2_{L^2(\Omega)} + \int_\Gamma |u|_\Gamma|^2 d\Gamma,$$

where $u|_\Gamma$ is the trace of u on Γ. Then there is a constant $C > 0$ such that

$$\|u\|_{H^1(\Omega)} \leq C\|u\|_{H^1(\Omega),\Gamma},$$

for all $u \in H^1(\Omega)$. Therefore, $\|u\|_{H^1(\Omega),\Gamma}$ is a norm equivalent to $\|u\|_{H^1(\Omega)}$ on $H^1(\Omega)$.

(ii) Let $V_\Gamma = \Big\{u \in H^1(\Omega),\ \int_\Gamma u d\Gamma = 0\Big\}$. Then$V_\Gamma$ is a closed subspace of $H^1(\Omega)$, and $|u|_{H^1(\Omega)}$ is a norm equivalent to $\|u\|_{H^1(\Omega)}$ on V_Γ.

Corollary: Let Ω an open and bounded domain, with Lipschitz-continuous boundary $\Gamma = \partial\Omega$. Then there is a positive constant C such that

$$\|u|_\Gamma\|_{L^2(\Gamma)} \leq C\|u\|_{H^1(\Omega)}.$$

Corollary: Over the space $H^2_0(\Omega)$, $\|\triangle u\|_{L^2(\Omega)}$ is a norm, equivalent to $\|u\|_{H^2(\Omega)}$.

- For s a real number, then $u \in H^s(R^n)$ if

$$(1 + |\xi|^2)^{s/2}\hat{u} \in L^2(R^n), \quad \xi \in R^n$$

(with $\hat{u}$ the Fourier transform of u).

We furnish $H^s(R^n)$ with the norm

$$\|u\|_s = \Big(\int_{R^n} (1 + |\xi|^2)^s |\hat{u}(\xi)|^2 d\xi\Big)^{1/2}.$$

For $s = m$ a non-negative integer, the space $H^s(R^n)$ coincides with the usual space $H^m(R^n)$.

- **Thm:** For $u \in H^1(\Omega)$, with $\Gamma = \partial\Omega$ of dimension $n-1$ and piecewise of class C^1, we can define $u|_\Gamma$ (the trace of u on Γ) as an element of $H^{1/2}(\Gamma)$.

Thm: For every $u_0 \in H^{1/2}(\Gamma)$, there is a $u \in H^1(\Omega)$ such that $u|_\Gamma = u_0$.

Note: For such set Γ, we can give a definition of $H^{1/2}(\Gamma)$ (with the aid of local maps defining Γ, see Lions-Magenes, Necas, Dautray-Lions, etc).

We also have another version of the Trace theorem:

Thm: Assume Ω is bounded and $\Gamma = \partial\Omega$ of class C^1. Then there exists a bounded linear operator

$$T : H^1(\Omega) \to L^2(\Omega)$$

such that

(i) $Tu = u|_\Gamma$ if $u \in H^1(\Omega) \cap C(\overline{\Omega})$

(ii)

$$\|Tu\|_{L^2(\Gamma)} \leq C\|u\|_{H^1(\Omega)},$$

for each $u \in H^1(\Omega)$, with constant C depending only on Ω.