Practice problems with partial solutions

Spring 2003, 269 C, Vese

[1] Write the differential equation

\[-\Delta u + u = f(x, y), \quad (x, y) \in \Omega\]
\[u = 1 \quad (x, y) \in \partial \Omega_1\]
\[\frac{\partial u}{\partial n} + u = x \quad (x, y) \in \partial \Omega_2,\]

where
\[\Omega = \{(x, y) \mid x^2 + y^2 < 1\},\]
\[\partial \Omega_1 = \{(x, y) \mid x^2 + y^2 = 1, \ x \leq 0\},\]
\[\partial \Omega_2 = \{(x, y) \mid x^2 + y^2 = 1, \ x > 0\},\]

in a weak variational form and describe a piecewise-linear Galerkin finite element approximation for the problem.

**Solution:** (I will discuss here only the weak formlation)

Let \(\Gamma_1 = \partial \Omega_1, \ \Gamma_2 = \partial \Omega_2, \ \Gamma = \partial \Omega.\) Change the notations: \((x, y) = (x_1, x_2).\)

Let \(V = \{v \in H^1(\Omega) : v = 0 \text{ on } \Gamma_1\}.\) Multiply the PDE by a test function and use Green’s formula. This gives the following:

\[-\int_{\Omega} v \Delta u dx + \int_{\Omega} u v dx = \int_{\Omega} f v dx,\]
\[\int_{\Omega} \nabla u \cdot \nabla v dx + \int_{\Gamma} v \frac{\partial u}{\partial n} ds = \int_{\Omega} f v dx.\]

We have on the boundary:

\[\int_{\Gamma} v \frac{\partial u}{\partial n} ds = \int_{\Gamma_1} v \frac{\partial u}{\partial n} ds + \int_{\Gamma_2} v \frac{\partial u}{\partial n} ds = 0 + \int_{\Gamma_2} v(x_1 - u) ds,\]

therefore the weak formulation is: Find \(u \in H^1(\Omega),\) with \(u = 1\) on \(\Gamma_1,\) such that

\[\int_{\Omega} (\nabla u \cdot \nabla v + uv) dx = \int_{\Omega} f v dx + \int_{\Gamma_2} v x dx,\]

for any \(v \in V.\)

Using the theorems from the class, it is possible to verify, without difficulty, that this problem satisfies the assumptions (i)-(iv) of the Lax-Milgram
Thm (exercise). Therefore, the problem has a unique solution (you may want to modify first the problem, by working with the new unknown variable \( w = u - 1 \) for the L-M lemma).

Other points to be discussed (left as exercise): For a FEM, start with a triangulation \( T_h \), define the space \( V_h \), give the discrete formulation, mention the basis functions, let \( v = \phi_j \) in the discrete weak problem, define the matrix \( A \) and the load vector \( b \), discuss properties of \( A \).

An error estimate gives us:

\[
|u - u_h|_{H^2(\Omega)} \leq C h |u|_{H^2(\Omega)}.
\]

[2] (a) Develop and describe the piecewise linear Galerkin finite element approximation of,

\[
\begin{align*}
-\nabla \cdot a(x) \nabla u + b(x) u &= f(x), & x &= (x_1, x_2) \in \Omega, \\
\frac{\partial u}{\partial x_1} + \frac{\partial u}{\partial x_2} + u &= 2, & x &\in \partial \Omega_1, \\
\frac{\partial u}{\partial x_1} + \frac{\partial u}{\partial x_2} + u &= 2, & x &\in \partial \Omega_2,
\end{align*}
\]

where

\[
\begin{align*}
\Omega &= \{ x | x_1 > 0, x_2 > 0, x_1 + x_2 < 1 \}, \\
\partial \Omega_1 &= \{ x | x_1 = 0, 0 \leq x_2 \leq 1 \} \cup \{ x | x_2 = 0, 0 \leq x_1 \leq 1 \}, \\
\partial \Omega_2 &= \{ x | x_1 > 0, x_2 > 0, x_1 + x_2 = 1 \}, \\
0 < a(x) &\leq A, 0 < b(x) \leq B.
\end{align*}
\]

(b) Justify the approximation by analyzing the appropriate bilinear and linear forms. Give a convergence estimate and quote the appropriate theorems for convergence.

**Solution:** (I will discuss here only the weak formulation)

Let \( \Gamma_1 = \partial \Omega_1, \Gamma_2 = \partial \Omega_2, \Gamma = \partial \Omega \).

The weak formulation is obtained as follows: let \( V = \{ v \in H^1(\Omega), v = 0 \text{ on } \Gamma_1 \} \). Multiply the PDE in \( \Omega \) by a test function \( v \in V \), integrate over \( \Omega \) and apply integration by parts:

\[
\begin{align*}
-\int_\Omega v \nabla \cdot a(x) \nabla u dx + \int_\Omega b(x) uv dx &= \int_\Omega f v dx, \\
\int_\Omega a(x) \nabla u \cdot \nabla v - \int_\Gamma a(x) v \nabla u \cdot \vec{n} ds + \int_\Omega b(x) uv dx &= \int_\Omega f v dx.
\end{align*}
\]
We have:

$$\int_{\Gamma} a(x)v \nabla u \cdot \vec{n} ds = \int_{\Gamma_1} a(x)v \nabla u \cdot \vec{n} ds + \int_{\Gamma_2} a(x)v \nabla u \cdot \vec{n} ds$$

$$= 0 + \int_{\Gamma_2} a(x)v \nabla u \cdot (1;1) ds = \int_{\Gamma_2} a(x)v(\frac{\partial u}{\partial x_1} + \frac{\partial u}{\partial x_2}) ds = \int_{\Gamma_2} a(x)v(2 - u) ds.$$

Therefore, the weak formulation is: Find $u \in H^1(\Omega)$, with $u = 2$ on $\Gamma_1$, such that

$$\int_{\Omega} (a(x) \nabla u \cdot \nabla v + (a(x) + b(x)) uv) dx = \int_{\Omega} f v dx + \int_{\Gamma_2} 2v ds,$$

for all $v \in V$.

To verify the assumptions (i)-(iv), use the fact that the functions $a$ and $b$ are strictly positive and bounded, and the theorems from the lecture. You may have to work with a new variable $w = u - 2$ for the L-M lemma.