Practice problems with partial solutions

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[1] Write the differential equation

$$
\begin{aligned}
-\triangle u + u = f(x,y), \quad & (x,y)\in\Omega \\
u = 1 \quad & (x,y)\in\partial\Omega_1 \\
\frac{\partial u}{\partial n} + u = x \quad & (x,y)\in\partial\Omega_2,
\end{aligned}
$$

where

$\Omega = \{(x,y)|\ x^2+y^2<1\}$,

$\partial\Omega_1 = \{(x,y)|\ x^2+y^2=1,\ x\le 0\}$,

$\partial\Omega_2 = \{(x,y)|\ x^2+y^2=1,\ x>0\}$,

in a weak variational form and describe a piecewise-linear Galerkin finite element approximation for the problem.

Solution: (I will discuss here only the weak fromulation)

Let $\Gamma_1 = \partial\Omega_1$, $\Gamma_2 = \partial\Omega_2$, $\Gamma = \partial\Omega$. Change the notations: $(x,y) = (x_1,x_2)$.

Let $V = \{v\in H^1(\Omega) : v = 0 \text{ on } \Gamma_1\}$. Multiply the PDE by a test function and use Green's formula. This gives the following:

$$-\int_\Omega v\triangle u dx + \int_\Omega uv dx = \int_\Omega fv dx,$$

$$\int_\Omega \nabla u\cdot\nabla v dx - \int_\Gamma v\frac{\partial u}{\partial n} ds = \int_\Omega fv dx.$$

We have on the boundary:

$$\int_\Gamma v\frac{\partial u}{\partial n}ds = \int_{\Gamma_1} v\frac{\partial u}{\partial n}ds + \int_{\Gamma_2} v\frac{\partial u}{\partial n}ds = 0 + \int_{\Gamma_2} v(x_1-u)ds,$$

therefore the weak formulation is: Find $u\in H^1(\Omega)$, with $u=1$ on Γ_1, such that

$$\int_\Omega(\nabla u\cdot\nabla v + uv)dx = \int_\Omega fv dx + \int_{\Gamma_2} vx ds,$$

for any $v\in V$.

Using the theorems from the class, it is possible to verify, without difficulty, that this problem satisfies the assumptions (i)-(iv) of the Lax-Milgram

Thm (exercise). Therefore, the problem has a unique solution (you may want to modify first the problem, by working with the new unknown variable $w = u - 1$ for the L-M lemma).

Other points to be discussed (left as exercise): For a FEM, start with a triangulation T_h, define the space V_h, give the discrete formulation, mention the basis functions, let $v = \phi_j$ in the discrete weak problem, define the matrix A and the load vector b, discuss properties of A.

An error estimate gives us:

$$|u - u_h|_{H^1(\Omega)} \leq Ch|u|_{H^2(\Omega)}.$$

[2] (a) Develop and describe the piecewise linear Galerkin finite element approximation of,

$$\begin{aligned} -\nabla \cdot a(x)\nabla u + b(x)u &= f(x), \quad & x = (x_1, x_2) \in \Omega, \\ u &= 2, & x \in \partial\Omega_1, \\ \frac{\partial u}{\partial x_1} + \frac{\partial u}{\partial x_2} + u &= 2, & x \in \partial\Omega_2, \end{aligned}$$

where

$\Omega = \{x|\ x_1 > 0,\ x_2 > 0,\ x_1 + x_2 < 1\}$,

$\partial\Omega_1 = \{x|\ x_1 = 0,\ 0 \leq x_2 \leq 1\} \cup \{x|\ x_2 = 0,\ 0 \leq x_1 \leq 1\}$,

$\partial\Omega_2 = \{x|\ x_1 > 0,\ x_2 > 0,\ x_1 + x_2 = 1\}$,

$0 < a \leq a(x) \leq A$, $0 < b \leq b(x) \leq B$.

(b) Justify the approximation by analyzing the appropriate bilinear and linear forms. Give a convergence estimate and quote the appropriate thoerems for convergence.

Solution: (I will discuss here only the weak fromulation)

Let $\Gamma_1 = \partial\Omega_1$, $\Gamma_2 = \partial\Omega_2$, $\Gamma = \partial\Omega$.

The weak formulation is obtained as follows: let $V = \{v \in H^1(\Omega), v = 0 \text{ on } \Gamma_1\}$. Multiply the PDE in Ω by a test function $v \in V$, integrate over Ω and apply integration by parts:

$$-\int_\Omega v\nabla \cdot a(x)\nabla u dx + \int_\Omega b(x)uvdx = \int_\Omega fvdx,$$

$$\int_\Omega a(x)\nabla u \cdot \nabla v - \int_\Gamma a(x)v\nabla u \cdot \vec{n}ds + \int_\Omega b(x)uvdx = \int_\Omega fvdx.$$

We have:

$$\int_{\Gamma} a(x)v\nabla u\cdot\vec{n}ds = \int_{\Gamma_1} a(x)v\nabla u\cdot\vec{n}ds + \int_{\Gamma_2} a(x)v\nabla u\cdot\vec{n}ds$$

$$= 0 + \int_{\Gamma_2} a(x)v\nabla u\cdot(1;1)ds = \int_{\Gamma_2} a(x)v(\frac{\partial u}{\partial x_1} + \frac{\partial u}{\partial x_2})ds = \int_{\Gamma_2} a(x)v(2-u)ds.$$

Therefore, the weak formulation is: Find $u \in H^1(\Omega)$, with $u = 2$ on Γ_1, such that

$$\int_{\Omega}(a(x)\nabla u\cdot\nabla v + (a(x)+b(x))uv)dx = \int_{\Omega} fvdx + \int_{\Gamma_2} 2vds,$$

for all $v \in V$.

To verify the assumptions (i)-(iv), use the fact that the functions a and b are strictly positive and bounded, and the theorems from the lecture. You may have to work with a new variable $w = u - 2$ for the L-M lemma.