Evolution problems of first order in time [1]

1.1 Function spaces

We are given a pair of real, separable Hilbert spaces V, H; we denote by
$((\ ,\ ))$ the scalar product, $\|\ \|$ the norm in V
$(\ ,\ )$ the scalar product, $|\ |$ the norm in H.

We suppose V is dense in H and we identify H with its dual H'. We also denote the duality between V' and V by $(\ ,\ )$.

1.2 The bilinear form $a(t;u,v)$, $t \in [0,T]$

For each $t \in [0,T]$, we are given a continuous bilinear form over $V \times V$ and we make the hypothesis:

(3.3) For every $u, v \in V$, the function $t \to a(t;u,v)$ is measurable and there is a constant $M = M(T) > 0$ (independent of $t \in]0,T[$, u, v) such that

$$|a(t;u,v)| \leq M\|u\|\|v\|$$

for all $u, v \in V$.

Def. Let $a, b \in R$. Then

$$W(V) = W(a,b;V,V') = \{u;\ u \in L^2(a,b;V),\ u' \in L^2(a,b;V')\}.$$

Proposition. This is a Hilbert space equipped with the norm

$$\|u\|_W = (\|u\|^2_{L^2(a,b;V)} + \|u'\|^2_{L^2(a,b;V')})^{1/2} = \Big(\int_a^b [\|u(t)\|^2_V + \|u'(t)\|^2_{V'}]dt\Big)^{1/2}.$$

We also assume (3.25) $a(t;u,u) \geq \alpha\|u\|^2_V$, for any $t \in [0,T]$, $u \in V$, and $u_0 \in H$, $f \in L^2(V')$.

Evolution Problem (P) Find u satisfying $u \in W(V)$,

$$\frac{d}{dt}(u(\cdot),v) + a(\cdot;u(\cdot),v) = (f(\cdot),v)$$

in the sense of distributions $\mathcal{D}'(]0,T[)$ for all $v \in V$, $u(0) = u_0$.

Remark. We have

$$\frac{d}{dt}(u(\cdot),v) = \Big(\frac{d}{dt}u(\cdot),v\Big),$$

for any $v \in V$.

Theorem 1. Then the solution of problem (P), if it exists, is unique.

Proof. Let u_1, u_2 be two distinct solutions of (P), then $w = u_1 - u_2$ satisfies $w \in W(V)$ and

$$\Big(\frac{dw}{dt}(\cdot),v\Big) + a(\cdot;w(\cdot),v) = 0,$$

[1] Following R. Dautray-J.-L. Lions, *Mathematical analysis and numerical methods for science and technology*, Volume 5, Evolution Problems I, Springer-Verlag, 1992.

for any $v \in V$, with $w(0) = 0$. Then by replacing v by $w(t)$ and integrating from 0 to t:

$$\frac{1}{2}|w(t)|^2 + \int_0^t a(s; w(s), w(s))ds = 0.$$

Since $a(\cdot; u, v)$ is $V-elliptic$, we have then

$$\frac{1}{2}|w(t)|^2 < 0 \Rightarrow w(t) = 0 \text{ for all } t \in [0, T].$$

Theorem 2. There exists a solution u to problem (P), and $u \in W(0, T; V, V')$.

Examples

1. Let Ω be an open and bounded subset of R^n, with boundary Γ, T finite, $V = H_0^1(\Omega)$, $H = L^2(\Omega)$, $V' = H^{-1}(\Omega)$. Let $\Omega_T = \Omega \times]0, T[$, $\Gamma_T = \Gamma \times]0, T[$.

The following problem

$$\frac{\partial u}{\partial t} - \triangle u = f, \quad u_{\Gamma_T} = 0, \quad u(\cdot, 0) = u_0 \text{ in } \Omega$$

has a unique solution using the bilinear form

$$a(t; u, v) = (\nabla u, \nabla v), \text{ for } t \in [0, T],$$

assuming $f \in L^2(0, T; H^{-1}(\Omega))$, $u_0 \in L^2(\Omega)$.

2. If we consider $V = H^1(\Omega)$ instead, $H = L^2(\Omega)$, and if a satisfies

$$a(t; u, u) + \lambda|u|^2 \geq \alpha\|u\|^2, \; t \in [0, T], \; u \in V,$$

then using a as in Example 1, we formally obtain that the Cauchy-Neumann problem has a unique solution:

$$\frac{\partial u}{\partial t} - \triangle u = f, \quad \frac{\partial u}{\partial n}|_{\Gamma_T} = 0, \quad u(\cdot, 0) = u_0 \text{ in } \Omega.$$

3. If f is such that, for any $v \in H^1(\Omega)$:

$$(f(t), v) = \int_\Omega f_0 v dx + \int_\Gamma f_1 v d\Gamma,$$

where $f_0 \in L^2(0, T; L^2(\Omega))$ and $f_1 \in L^2(0, T; H^{-1/2}(\Gamma))$, then $f \in L^2(0, T; V')$ and the corresponding problem is

$$\frac{\partial u}{\partial t} - \triangle u = f, \quad \frac{\partial u}{\partial n}|_{\Gamma_T} = f_1, \quad u(\cdot, 0) = u_0 \text{ in } \Omega.$$

4. Mixed Dirichlet-Neumann BC can be considered.