

**269C, Vese**  
**Useful results 2**

- For  $s$  a real number, then  $u \in H^s(\mathbb{R}^n)$  if

$$(1 + |\xi|^2)^{s/2} \hat{u} \in L^2(\mathbb{R}^n), \quad \xi \in \mathbb{R}^n$$

(with  $\hat{u}$  the Fourier transform of  $u$ ).

We furnish  $H^s(\mathbb{R}^n)$  with the norm

$$\|u\|_s = \left( \int_{\mathbb{R}^n} (1 + |\xi|^2)^s |\hat{u}(\xi)|^2 d\xi \right)^{1/2}.$$

For  $s = m$  a non-negative integer, the space  $H^s(\mathbb{R}^n)$  coincides with the usual space  $H^m(\mathbb{R}^n)$ .

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**Thm:** For  $u \in H^1(\Omega)$ , with  $\Gamma = \partial\Omega$  of dimension  $n - 1$  and piecewise of class  $C^1$ , we can define  $u|_\Gamma$  (the trace of  $u$  on  $\Gamma$ ) as an element of  $H^{1/2}(\Gamma)$ .

**Thm:** For every  $u_0 \in H^{1/2}(\Gamma)$ , there is a  $u \in H^1(\Omega)$  such that  $u|_\Gamma = u_0$ .

**Note:** For such set  $\Gamma$ , we can give a definition of  $H^{1/2}(\Gamma)$  (with the aid of local maps defining  $\Gamma$ , see Lions-Magenes, Necas, Dautray-Lions, etc).

We also have another version of the Trace theorem:

**Thm:** Assume  $\Omega$  is bounded and  $\Gamma = \partial\Omega$  of class  $C^1$ . Then there exists a bounded linear operator

$$T : H^1(\Omega) \rightarrow L^2(\Gamma)$$

such that

- (i)  $Tu = u|_\Gamma$  if  $u \in H^1(\Omega) \cap C(\overline{\Omega})$
- (ii)

$$\|Tu\|_{L^2(\Gamma)} \leq C \|u\|_{H^1(\Omega)},$$

for each  $u \in H^1(\Omega)$ , with constant  $C$  depending only on  $\Omega$ .

**Additional results, remarks, exercises**

(see “Finite Elements, Mathematical Aspects”, Vol. IV, J. Tinsley Oden and Graham F. Carey, Prentice Hall, 1983)

**Thm.** *The Generalized Lax-Milgram Theorem:* Let  $H$  and  $G$  be real Hilbert spaces and let  $B(\cdot, \cdot)$  denote a bilinear form on  $H \times G$  which has the following properties:

(i)  $B(\cdot, \cdot)$  is continuous; that is, there exists a constant  $M > 0$  such that

$$|B(u, v)| \leq M \|u\|_H \|v\|_G, \text{ for any } u \in H, v \in G.$$

(ii)  $B(\cdot, \cdot)$  is coercive in the sense that there exists a constant  $\alpha$  such that

$$\inf_{u \in H, \|u\|_H=1} \sup_{v \in G, \|v\|_G \leq 1} |B(u, v)| \geq \alpha > 0.$$

(iii) For every  $v \neq 0$  in  $G$ ,

$$\sup_{u \in H} |B(u, v)| > 0.$$

Then there exists a unique  $u^* \in H$  such that

$$B(u^*, v) = F(v), \text{ for all } v \in G,$$

wherein  $F \in G'$ . Moreover,

$$\|u^*\|_H \leq \frac{1}{\alpha} \|F\|_{G'}.$$

If  $H = G$ , then (ii) and (iii) can be replaced by the simpler condition:

(iv) There exists an  $\alpha > 0$  such that

$$B(u, u) \geq \alpha \|u\|_H^2, \text{ for all } u \in H.$$

(in this case, we say that  $B$  is  $H$  – coercive).

Here  $G'$  denotes the dual of  $G$ , the space of continuous linear forms  $F$  on  $H$ .

## Mixed Methods

Notations and conventions:

- $H, Q$  are real Hilbert spaces with norms  $\|\cdot\|_H$  and  $\|\cdot\|_Q$  respectively.
- $a : H \times H$  is a symmetric, continuous bilinear form so that  $M > 0$  exists such that

$$|a(u, v)| \leq M\|u\|_H\|v\|_H, \text{ for all } u, v \in H.$$

- $b : H \times Q$  is a continuous bilinear form so that  $m > 0$  exists such that

$$|b(u, q)| \leq m\|u\|_H\|q\|_Q, \text{ for all } u \in H, q \in Q.$$

- $f : H \rightarrow R$  is a continuous linear form on  $H$ .
- $g : Q \rightarrow R$  is a continuous linear form on  $Q$ .

We consider the weak variational formulation

$$a(u, v) + b(v, p) = f(v), \text{ for all } v \in H \quad (1)$$

$$b(u, q) = g(q) \text{ for all } q \in Q. \quad (2)$$

This corresponds to the abstract problem (in distributional sense)

$$Au + B^*p = f \text{ in } H',$$

$$Bu = g \text{ in } Q'.$$

We also define:

$$\ker B = \{v \in H | b(v, q) = 0 \text{ for all } q \in Q\},$$

$$\ker B^* = \{q \in Q | b(v, q) = 0 \text{ for all } v \in H\},$$

and the quotient space

$$Z = Q/\ker B^*.$$

The quotient space  $Z$  consists of equivalence classes  $[p]$  of the form

$$[p] = \{q \in Q | p - q \in \ker B^*\}.$$

We also assume that  $g \in Rg(B)$ .

Note that  $Bu = g$  is a constraint, corresponding to an unknown Lagrange multiplier  $p$  in (1).

Problem (1)-(2) can be substituted by the following equivalent problem: introduce a bilinear form  $B(\cdot, \cdot)$  defined on the product space  $H \times Q$ ,

$B : (H \times Q) \times (H \times Q) \rightarrow R$ , by

$$B((u, p), (v, q)) = a(u, v) + b(v, p) + b(u, q)$$

and the linear form  $F : H \times Q \rightarrow R$ ,

$$F((v, q)) = f(v) + g(q).$$

Then (1)-(2) can be written: find  $(u, p) \in H \times Q$  s.t.

$$B((u, p), (v, q)) = F((v, q)), \text{ for all } (v, q) \in H \times Q. \quad (3)$$

Also impose conditions: there is  $\alpha_0 > 0$  s.t.

$$\alpha_0 \|u_0\|_H \leq \sup_{v_0 \in \ker B - \{0\}} \frac{|a(u_0, v_0)|}{\|v_0\|_H}, \text{ for all } u_0 \in \ker B, \quad (4)$$

and there is  $\beta > 0$  s.t.

$$\beta \inf_{p_0 \in \ker B^*} \|p + p_0\|_Q = \beta \| [p] \|_Z \leq \sup_{v \in H - \{0\}} \frac{|b(v, p)|}{\|v\|_H}, \text{ for all } p \in Q. \quad (5)$$

Conditions (4) and (5) are equivalent with: there is  $\alpha > 0$  s.t. for all  $(u, p) \in H \times Q$ ,

$$\alpha (\|u\|_H + \| [p] \|_Z) \leq \sup_{(v, q) \in H \times Q, (v, q) \neq (0, 0)} \frac{|a(u, v) + b(v, p) + b(u, q)|}{\|v\|_H + \|q\|_Q}.$$

**Thm.** Let conditions (4) and (5) hold for the continuous bilinear forms  $a(\cdot, \cdot)$  and  $b(\cdot, \cdot)$  defined in the beginning. Then there exists a unique solution  $(u, [p]) \in H \times Z$ ,  $Z = Q/\ker B^*$ , of problem (3) with  $g \in \text{Rg}(B)$ . The Lagrange multiplier  $p$  is then unique up to an arbitrary element of  $\ker B^*$ .

**Problems:**

- Consider the Neumann problem

$$-\Delta u = 0 \text{ in } \Omega,$$

$$\frac{\partial u}{\partial \vec{n}} = g \text{ on } \partial\Omega,$$

where  $\Omega$  is sufficiently smooth and  $g \in L^2(\partial\Omega)$ .

- (a) Give a weak variational formulation of the problem.
- (b) Give a condition on  $g$  necessary to guarantee the existence of a solution to this problem.
- (c) Give a condition on  $u$  necessary to guarantee the existence and uniqueness of a solution to this problem.

- Apply the above mixed method to the stationary Stokes equations.