**269C, Vese**
**Useful results**

Notations:
For $u \in H^m(\Omega)$, let

$$\|u\|_{H^m(\Omega)} = \Big(\sum_{|\alpha|\le m} \int_\Omega |D^\alpha u(x)|^2 dx\Big)^{1/2} = \Big(\sum_{|\alpha|\le m} \|D^\alpha u\|^2_{L^2(\Omega)}\Big)^{1/2},$$

$$|u|_{H^m(\Omega)} = \Big(\sum_{|\alpha|= m} \int_\Omega |D^\alpha u(x)|^2 dx\Big)^{1/2} = \Big(\sum_{|\alpha|= m} \|D^\alpha u\|^2_{L^2(\Omega)}\Big)^{1/2}.$$

**Thm.** (Poincaré's Inequality for $H^1_0(\Omega)$)
Let $\Omega$ be an open and bounded set in $R^n$. Then there is a positive constant $C = C(\Omega)$ such that, for all $u \in H^1_0(\Omega)$, we have Poincaré inequality:

$$\|u\|^2_{L^2(\Omega)} \le C\|\nabla u\|^2_{L^2(\Omega)}.$$

**Corollary:** Let $m > 0$ be a positive integer, and let $\Omega$ be an open and bounded set in $R^n$. Then for $u \in H^m_0(\Omega)$, we have

$$\|u\|^2_{L^2(\Omega)} \le C_m C^m \sum_{|\alpha|=m} \|D^\alpha u\|^2_{L^2(\Omega)},$$

$C_m$ =constant, and $C$ is the constant from the previous theorem.

**Corollary:** (same assumptions on $\Omega$). $|u|_{H^m(\Omega)}$ is a norm on $H^m_0(\Omega)$, equivalent to the norm $\|u\|_{H^m(\Omega)}$.

**Thm.** Let $\Omega$ be a bounded connected open set in $R^n$, with sufficiently regular boundary. Then we have for $u \in H^1(\Omega)$, such that $\int_\Omega u(x)dx = 0$,

$$|u|^2_{L^2(\Omega)} \le P(\Omega)\|\nabla u\|^2_{L^2(\Omega)}.$$

More generally, we have for $u \in H^1(\Omega)$

$$|u|^2_{L^2(\Omega)} \le P(\Omega)\|\nabla u\|^2_{L^2(\Omega)} + \frac{1}{|\Omega|}\Big|\int_\Omega u(x)dx\Big|^2.$$

**Corollary:** $|u|_{H^1(\Omega)} = \|\nabla u\|_{L^2(\Omega)}$ is a norm equivalent with the norm $\|u\|_{H^1(\Omega)}$ on the sub-space $V_0$ (closed in $H^1(\Omega)$) defined by:

$$V_0 = \{u \in H^1(\Omega) : \int_\Omega u(x)dx = 0\}.$$

**Corollary:** Let $\Omega$ be a bounded connected open set in $R^n$, with sufficiently regular boundary $\Gamma$. Suppose $\Gamma = \Gamma_1 \cup \Gamma_2$ with length (area) of $\Gamma_2 > 0$. Let

$$V_{\Gamma_2} = \{u \in H^1(\Omega) : \ u|_{\Gamma_2} = 0\}.$$

Then $V_{\Gamma_2}$ is a closed sub-space of $H^1(\Omega)$ and $|u|_{H^1(\Omega)} = \|\nabla u\|_{L^2(\Omega)}$ is a norm equivalent with the norm $\|u\|_{H^1(\Omega)}$ on the sub-space $V_{\Gamma_2}$.

**Remark:**

(i) Suppose that $\Omega$ is a bounded connected open set in $R^n$ which is "very regular" ($\Gamma = \partial\Omega$ is a $n-1$ dimensional manifold of class $C^\infty$ and $\Omega$ locally on one side of $\Gamma$). For $u \in H^1(\Omega)$, let

$$\|u\|^2_{H^1(\Omega),\Gamma} = \|\nabla u\|^2_{L^2(\Omega)} + \int_\Gamma |u|_\Gamma|^2 d\Gamma,$$

where $u|_\Gamma$ is the trace of $u$ on $\Gamma$. Then there is a constant $C > 0$ such that

$$\|u\|_{H^1(\Omega)} \leq C\|u\|_{H^1(\Omega),\Gamma},$$

for all $u \in H^1(\Omega)$. Therefore, $\|u\|_{H^1(\Omega),\Gamma}$ is a norm equivalent to $\|u\|_{H^1(\Omega)}$ on $H^1(\Omega)$.

(ii) Let $V_\Gamma = \left\{u \in H^1(\Omega), \ \int_\Gamma u d\Gamma = 0\right\}$. Then$V_\Gamma$ is a closed subspace of $H^1(\Omega)$, and $|u|_{H^1(\Omega)}$ is a norm equivalent to $\|u\|_{H^1(\Omega)}$ on $V_\Gamma$.

**Corollary:** Let $\Omega$ an open and bounded domain, with Lipschitz-continuous boundary $\Gamma = \partial\Omega$. Then there is a positive constant $C$ such that

$$\|u|_\Gamma\|_{L^2(\Gamma)} \leq C\|u\|_{H^1(\Omega)}.$$

**Corollary:** Over the space $H_0^2(\Omega)$, $\|\triangle u\|_{L^2(\Omega)}$ is a norm, equivalent to $\|u\|_{H^2(\Omega)}$.