HW #3, 269C, Vese

Due on Monday, May 17

1.

(a) Give a weak variational formulation of the problem

$$\frac{d^4u}{dx^4} = f \quad \text{for} \quad 0 < x < 1,$$
$$u(0) = u''(0) = u'(1) = u'''(1) = 0,$$

and show that the conditions (i)-(iv) are satisfied. Which boundary conditions are essential and which are natural ?

(b) Solve the same problem with the following alternative boundary conditions:

$$u(0) = -u''(0) + \gamma u'(0) = 0, \quad u(1) = u''(1) + \gamma u'(1) = 0,$$

where γ is a positive constant.

2. Give a weak variational formulation of the Neumann problem

$$-\triangle u + b(x)u = f \quad \text{in } \Omega,$$
$$\frac{\partial u}{\partial \vec{n}} = g \quad \text{on } \Gamma,$$

with the following assumptions on the functions b, f and g:

$$b \in L^\infty(\Omega), \; b \geq b_0 > 0 \text{ a.e. on } \Omega, \; f \in L^2(\Omega), \; g \in L^2(\Gamma),$$

for some constant b_0. Check if the conditions (i)-(iv) are satisfied.

3. Give a weak variational formulation of the Robin's problem

$$-\triangle u = f \;\text{ in } \Omega, \quad \gamma u + \frac{\partial u}{\partial n} = g \text{ on } \Gamma,$$

where γ is a constant. When are conditions (i)-(iv) satisfied ?

4. Consider the Neumann problem

$$-\triangle u = f \text{ in } \Omega,$$
$$\frac{\partial u}{\partial n} = g \text{ on } \Gamma,$$
$$\int_\Omega u(x)dx = 0.$$

(a) Why condition "$\int_\Omega u(x)dx = 0$" was added here ?

(b) Give a variational formulation of the problem, and prove that the conditions (i)-(iv) are satisfied, under the "usual" assumptions on f and g.

5. Consider the inhomogeneous Dirichlet problem

$$-\triangle u + b(x)u = f \text{ in } \Omega, \qquad u = u_0 \text{ on } \Gamma,$$

where $b \in L^\infty(\Omega)$, $b(x) \geq 0$ a.e. in Ω, $f \in L^2(\Omega)$, and $u_0|_\Gamma$ is the trace of a $u_0 \in H^1(\Omega)$. To solve this problem by Lax-Milgram, modify it into a homogeneous Dirichlet problem, and analyze the resulting weak variational problem.