MATH 269B: Final assignment (due at the end of the week of finals)

You can leave it with Jacquie Bowens in MS 7619, or slide it under the door of my office, or in my mailbox.

[**1**] Consider the heat equation

$$u_t = b u_{xx},$$

to be solved for $t > 0$, with smooth initial data $u(x, 0) = u_0(x)$ and real x.

(a) Under what condition on the parameter b is this a well-posed problem ? Explain.

(b) For the same problem, consider the scheme

$$\frac{u_j^{n+1} - u_j^n}{\triangle t} = b\Big[\theta \triangle_x^2 u_j^{n+1} + (1-\theta)\triangle_x^2 u_j^n\Big],$$

where

$$\triangle_x^2 u_j^n = \frac{u_{j+1}^n - 2u_j^n + u_{j-1}^n}{h^2}.$$

(i) What well known schemes do we obtain for $\theta = 0$, $\theta = 1$ and $\theta = \frac{1}{2}$?
(ii) Show that for $\frac{1}{2} \leq \theta \leq 1$, this is an unconditionally stable scheme.

[**2**] For the equation

$$u_{tt} + u_t = u_{xx} + u_x$$

to be solved for $t > 0$, with smooth initial data

$$u(x, 0) = u_0(x), \quad u_t(x, 0) = u_1(x)$$

(a) Restate this problem as an equivalent system of two first order equations (first order in both time and space), using a "factored" form of the initial second order equation. Thus obtain an equivalent system of the form

$$\vec{U}_t = A\vec{U}_x + B\vec{U},$$

with A and B two 2x2 matrices (note that the term $B\vec{U}$ will be a lower order term).

(b) Give a convergent finite difference approximation of your choice to this first order system. Justify your answers.

[**3**] To solve

$$u_t + a u_x = 0 \quad \text{for } t > 0, \; 0 \leq x \leq 1,$$

$u(x, 0) = \phi(x)$ smooth, u periodic in x, $u(x + 1, t) = u(x, t)$, we use:

$$\frac{1}{2\triangle t}\Big[(v_j^{n+1} + v_{j+1}^{n+1}) - (v_j^n + v_{j+1}^n)\Big] + \frac{a}{2\triangle x}\Big[v_{j+1}^{n+1} - v_j^{n+1} + v_{j+1}^n - v_j^n\Big] = 0.$$

For what values of $\frac{\triangle t}{\triangle x}$, if any, does this converge ? At what rate ? Explain your answers.

(Recall that for smooth initial data, the order of accuracy of the scheme gives the order of accuracy of the solution.)

(Note that sometimes the computation of $|g(\theta)|^2$ is simplified if you use $e^{i\theta} = (e^{i\theta/2})^2$, $e^{-i\theta} = (e^{-i\theta/2})^2$, and $1 = e^{i\theta/2}e^{-i\theta/2}$. These could have been used for the angled derivative method.)

Optional problems

[**4**] Consider the differential equation

$$u_t = u_{xx} + bu_{xy} + u_{yy} \quad \text{for } t > 0,\ 0 < x < 1,\ 0 < y < 1,$$

with $u = 0$ on the boundary, and $u(0, x, y) = \phi(x, y)$, a smooth function.

(a) For what values of b can you obtain a convergent, unconditionally stable finite difference scheme ?

(b) Construct such a scheme. Explain your answers.

Note that this is a particular case of the more general problem, given next.

[**5**] Consider the PDE

$$u_t = Au_{xx} + 2Bu_{xy} + Cu_{yy},$$

with associated smooth initial and boundary conditions.

(a) For what constants A, B, C is this a well-posed parabolic equation ? (answer: if $A > 0,\ C > 0,\ B^2 < AC$; explain).

(b) Apply the Θ-method to this problem (as outlined below).

Let $\Phi^n_{j,l} = \frac{A}{(\triangle x)^2}(u^n_{j+1,l} - 2u^n_{j,l} + u^n_{j-1,l}) + \frac{B}{2\triangle x \triangle y}(u^n_{j+1,l+1} - u^n_{j-1,l+1} - u^n_{j+1,l-1} + u^n_{j-1,l-1}) + \frac{C}{(\triangle y)^2}(u^n_{j,l+1} - 2u^n_{j,l} + u^n_{j,l-1})$.

The general Θ-method, for $0 \le \Theta \le 1$, is

$$\frac{u^{n+1}_{j,l} - u^n_{j,l}}{\triangle t} = \Theta \Phi^{n+1}_{j,l} + (1 - \Theta)\Phi^n_{j,l}.$$

(c) Obtain the stability conditions of the scheme, function of Θ.

Hint:

- Use $u^n_{j,l} = g^n e^{ij\theta_1} e^{il\theta_2}$ as usual.

- Compute the amplification factor g of the scheme, and write it in the form $g = \frac{1+(1-\Theta)\Psi}{1-\Theta\Psi}$, for $\Psi = \Psi(\theta_1, \theta_2)$ to be determined.

- Notice that Ψ is a real analytic periodic function of (θ_1, θ_2), therefore $g(\theta_1, \theta_2)$ is also real.

- To see under what conditions $|g(\theta_1, \theta_2)| \le 1$, find first the minimum and maximum values of Ψ (to find the extema of Ψ, write a linear system of two equations obtained from $\frac{\partial \Psi}{\partial \theta_1} = \frac{\partial \Psi}{\partial \theta_2} = 0$; consider the unknowns $\sin\theta_1$ and $\sin\theta_2$).

- Using the lower and upper bounds of Ψ, you will obtain the stability conditions of the scheme (function of Θ).