Review and practice problems for the midterm

[**1**] Consider the one-way wave equation $u_t + u_x = 0$ for $t > 0$ and $x \in R$, with the initial condition $u(x, 0) = u_0(x)$.

(a) Give the exact solution of the equation.

(b) Show that the leapfrog scheme with $\lambda = \frac{k}{h} = 1$ applied to this equation gives the exact solution (let v^0_m and v^1_m be given by the exact solution at $t = 0$ and $t = k$, for all $m \in Z$).

[**2**] Consider the system of partial differential equations

$$(1) \qquad \begin{cases} u_t + u_x + v_x = 0, & u(x,0) = u_0(x), \\ v_t + u_x - v_x = 0, & v(x,0) = v_0(x). \end{cases}$$

(a) Write the system in matrix form $U_t + AU_x = \vec{0}$, with $U(x,t) = \begin{pmatrix} u(x,t) \\ v(x,t) \end{pmatrix}$, and find the matrix A.

(b) Is this a hyperbolic system ? Explain.

(c) Reduce, if possible, the system (1) to a set of independent scalar hyperbolic equations.

(d) Obtain the exact solution of the system (1).

(e) Let $V^n_m \approx \begin{pmatrix} u(x_m, t_n) \\ v(x_m, t_n) \end{pmatrix}$. Consider the FTCS scheme applied to the system (1) in matrix form:

$$(2) \qquad \frac{1}{k}(V^{n+1}_m - V^n_m) + \frac{1}{2h}A(V^n_{m+1} - V^n_{m-1}) = \vec{0}.$$

Find the nonsingular amplification matrix G for this scheme by substituting $V^n_m = G^n e^{im\theta}$ into the difference equation (2).

(f) Show the following general statement:

Assume that A is a diagonalizable matrix (i.e. there is a nonsingular matrix P such that $PAP^{-1} = \Lambda_A$ is diagonal). If $G = \Phi(A)$, where $\Phi(A) = c_0 I + c_1 A + c_2 A^2$ is a polynomial of A, whith I the identity matrix, and c_0, c_1, c_2 constants, then G is also diagonalizable, with the same matrix P.

Moreover, the diagonal matrix Λ_G of eigenvalues of G can easily be obtained function of Λ_A (show that $\Lambda_G = \Phi(\Lambda_A)$).

Note 1. The above statement is true for any polynomial or rational function Φ of A (you do not have to prove Note 1).

Note 2. Assume that A and G are as in statement (f), with A a constant matrix, and G the amplification matrix. Then the von Neumann condition is necessary and sufficient for stability (hint: write $G = P^{-1}\Lambda_G P$, $\|G^n\|_2 = \|(P^{-1}\Lambda_G P)^n\|_2 \leq \|P^{-1}\|_2\|P\|_2\|\Lambda_G^n\|_2$, and use the general formula $\|M\|_2 = \rho(M^*M)$, where M is an nxn matrix and $\rho(M^*M)$ is the spectral radius of M^*M).

(g) Apply the previous statements to the particular matrices A and G from (a)-(e) and deduce that the scheme (2) is unstable if $\lambda = \frac{k}{h}$ is constant.

Def. (order of accuracy for homogeneous equations) If $P_{k,h}\phi = O(h^r)$ for each formal solution to $P\phi = 0$, then the scheme is accurate of order r, provided that $k = \Lambda(h)$.

[3] Consider the system of equations

$$\begin{pmatrix} u \\ v \end{pmatrix}_t = A \begin{pmatrix} u \\ v \end{pmatrix}_x ; \qquad A = \begin{pmatrix} 1 & \alpha \\ 0 & 1 \end{pmatrix},$$

which is approximated by the Lax-Friedrichs scheme, $V_j^n \approx (u(x_j, t_n), v(x_j, t_n))$,

$$V_j^{n+1} = \frac{1}{2}(V_{j+1}^n + V_{j-1}^n) + \frac{\triangle t}{2\triangle x} A(V_{j+1}^n - V_{j-1}^n).$$

Determine the order of accuracy of this scheme and its stability properties depending on whether $\alpha \neq 0$ or $\alpha = 0$.