Review and practice problems for the midterm

[1] Consider the one-way wave equation $u_t + u_x = 0$ for $t > 0$ and $x \in R$, with the initial condition $u(x, 0) = u_0(x)$.

(a) Give the exact solution of the equation.
(b) Show that the leapfrog scheme with $\lambda = \frac{k}{h} = 1$ applied to this equation gives the exact solution (let $v^0_m$ and $v^1_m$ be given by the exact solution at $t = 0$ and $t = k$, for all $m \in Z$).

[2] Consider the system of partial differential equations

\[
\begin{align*}
&u_t + u_x + v_x = 0, & u(x, 0) = u_0(x), \\
v_t + u_x - v_x = 0, & v(x, 0) = v_0(x).
\end{align*}
\]

(a) Write the system in matrix form $U_t + AU_x = 0$, with $U(x,t) = \begin{pmatrix} u(x,t) \\ v(x,t) \end{pmatrix}$, and find the matrix $A$.
(b) Is this a hyperbolic system? Explain.
(c) Reduce, if possible, the system (1) to a set of independent scalar hyperbolic equations.
(d) Obtain the exact solution of the system (1).
(e) Let $V^n_m \approx \begin{pmatrix} u(x_m, t_n) \\ v(x_m, t_n) \end{pmatrix}$. Consider the FTCS scheme applied to the system (1) in matrix form:

\[
\frac{1}{k} (V_{m+1}^n - V_m^n) + \frac{1}{2h} A(V_{m+1}^n - V_{m-1}^n) = 0.
\]

Find the nonsingular amplification matrix $G$ for this scheme by substituting $V^n_m = G^n e^{im\theta}$ into the difference equation (2).
(f) Show the following general statement:

Assume that $A$ is a diagonalizable matrix (i.e. there is a nonsingular matrix $P$ such that $PAP^{-1} = \Lambda_A$ is diagonal). If $G = \Phi(A)$, where $\Phi(A) = c_0 I + c_1 A + c_2 A^2$ is a polynomial of $A$, whith $I$ the identity matrix, and $c_0$, $c_1$, $c_2$ constants, then $G$ is also diagonalizable, with the same matrix $P$.

Moreover, the diagonal matrix $\Lambda_G$ of eigenvalues of $G$ can easily be obtained function of $\Lambda_A$ (show that $\Lambda_G = \Phi(\Lambda_A)$).

Note 1. The above statement is true for any polynomial or rational function $\Phi$ of $A$ (you do not have to prove Note 1).
Note 2. Assume that $A$ and $G$ are as in statement (f), with $A$ a constant matrix, and $G$ the amplification matrix. Then the von Neumann condition is necessary and sufficient for stability (hint: write $G = P^{-1} \Lambda G P$, $\|G^n\|_2 = \|(P^{-1} \Lambda G P)^n\|_2 \leq \|P^{-1}\|_2\|P\|_2\|\Lambda^n\|_2$, and use the general formula $\|M\|_2 = \rho(M^*M)$, where $M$ is an $nxn$ matrix and $\rho(M^*M)$ is the spectral radius of $M^*M$).

(g) Apply the previous statements to the particular matrices $A$ and $G$ from (a)-(e) and deduce that the scheme (2) is unstable if $\lambda = \frac{k}{n}$ is constant.

Def. (order of accuracy for homogeneous equations) If $P_{k,h}\phi = O(h^r)$ for each formal solution to $P\phi = 0$, then the scheme is accurate of order $r$, provided that $k = \Lambda(h)$.

[3] Consider the system of equations

$$\begin{pmatrix} u \\ v \end{pmatrix}_t = A \begin{pmatrix} u \\ v \end{pmatrix}_x; \quad A = \begin{pmatrix} 1 & \alpha \\ 0 & 1 \end{pmatrix},$$

which is approximated by the Lax-Friedrichs scheme, $V^n_j \approx (u(x_j, t_n), v(x_j, t_n))$,

$$V^{n+1}_j = \frac{1}{2}(V^n_{j+1} + V^n_{j-1}) + \frac{\Delta t}{2\Delta x} A(V^n_{j+1} - V^n_{j-1}).$$

Determine the order of accuracy of this scheme and its stability properties depending on whether $\alpha \neq 0$ or $\alpha = 0$. 

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