Math 269B. Instructor: Luminita Vese. Teaching Assistant: Michael Puthawala.
Homework #5 Due date: Friday, February 24

[1] Consider the parabolic model problem $u_t = u_{xx}$ for $t > 0$, $0 < x < 1$, with the B.C. $u(0,t) = u(1,t) = 0$ for $t > 0$, and with the I.C. $u(x,0) = u^0(x)$, for $0 \le x \le 1$.

(a) Using separation of variables $u(x,t) = f(x)g(t)$ and solving for f and g, we can obtain an exact analytic solution of the problem function of u^0, which is:

$$u(t,x) = \sum_{m=1}^{\infty} a_m e^{-(m\pi)^2 t} \sin(m\pi x),$$

with coefficients a_m obtained from the initial condition:

$$a_m = 2\int_0^1 u^0(x)\sin(m\pi x)dx.$$

A first few terms of the series solution can be used to approximate very well the exact solution for small values of t (to be used in part (b)).

Show that the above formula for $u(t,x)$ is indeed the exact solution of this initial and boundary value problem (or derive the solution using separation of variables; in the derivation, use the constant $-k^2$ when arriving to $fg' = f''g$ since the solution is bounded in time, and then set $k = m\pi$ with m positive integer to select the solutions that satisfy the B.C.).

(b) Assume that $u^0(x) = \begin{cases} 2x, & \text{if } 0 \le x \le \frac{1}{2} \\ 2-2x, & \text{if } \frac{1}{2} \le x \le 1. \end{cases}$

Using $\triangle x = 0.05$ and for two different time steps $\triangle t = 0.0012$ and $\triangle t = 0.0013$, compute a numerical approximation using the explicit scheme (forward in time, central in space). Implement the algorithm and plot two sets of results for each time step at $t = 0$ and after 1, 25 and 50 time steps. Compare the numerical approximations with the exact solution. You should turn in the numerical scheme, the code and plots of results. What can you observe ?

[**2**] Show that a scheme for $u_t = bu_{xx}$ of the form

$$v_m^{n+1} = \alpha v_m^n + \frac{1-\alpha}{2}(v_{m+1}^n + v_{m-1}^n),$$

with α constant as h and k tend to zero, is consistent with the heat equation only if $\alpha = 1 - 2b\mu$.

[**3**] Consider the Crank-Nicolson scheme for $u_t = bu_{xx}$:

$$\frac{v_m^{n+1} - v_m^n}{k} = \frac{1}{2}b\frac{v_{m+1}^{n+1} - 2v_m^{n+1} + v_{m-1}^{n+1}}{h^2} + \frac{1}{2}b\frac{v_{m+1}^n - 2v_m^n + v_{m-1}^n}{h^2}.$$

(a) Show that this scheme satisfies the *maximum norm stability*

$$\|v^{n+1}\|_\infty \leq \|v^n\|_\infty$$

for all solutions if $b\mu \leq 1$. *Hint:* Show that if $v_{m'}^{n+1}$ is the largest value of v_m^{n+1}, then

$$v_{m'}^{n+1} \leq -\frac{b\mu}{2} v_{m'-1}^{n+1} + (1 + b\mu) v_{m'}^{n+1} - \frac{b\mu}{2} v_{m'+1}^{n+1} \leq \|v^n\|_\infty.$$

(b) Find the order of the scheme.