Math 269B Homework #4 Due date: Friday, February 17
Instructor: Luminita Vese. Teaching Assistant: Michael Puthawala.

[**1**] Consider the one-way wave equation

$$\frac{\partial u}{\partial t} + \frac{\partial u}{\partial x} = 0.$$

Analyze the stability and the order of accuracy of the following *angled derivative method*

$$u_j^{n+2} = (1-2\lambda)(u_j^{n+1} - u_{j-1}^{n+1}) + u_{j-1}^n, \quad n \geq 0,$$

with $\lambda = \frac{\triangle t}{\triangle x}$.

[**2**] *Computational assignment.* Consider the one-way wave equation

$$\begin{aligned} &\frac{\partial u}{\partial t} + \frac{\partial u}{\partial x} = 0, \quad t > 0, \quad 0 < x < 2, \\ &u(x,0) = e^{-100(x-1/2)^2}\sin(20\pi x), \\ &u(0,t) = u(2,t) = 0, \quad t \geq 0. \end{aligned}$$

Find approximations of the exact solution at $t = 1$, using $\triangle x = \frac{1}{80}$ and $\lambda = \frac{\triangle t}{\triangle x} = \frac{2}{3}$, by three methods: the leapfrog method, the Lax-Wendroff method, and the angled derivative method.

Comment and compare the obtained approximations, function of accuracy, dissipation, dispersion, conservation (a non-dissipative scheme is called conservative). Give and plot the exact solution also at $t = 1$.

[**3**] *Computational assignment.* Consider now the same equation as in [2], but with the initial condition

$$u(x,0) = \begin{cases} 1, & \text{if } \frac{1}{4} \leq x < \frac{3}{4}, \\ 0, & \text{otherwise.} \end{cases}$$

Find numerical approximations using four methods: the three methods used in [2] and the forward-in-time backward-in-space method (FTBS). In all cases, take $\triangle x = \frac{1}{100}$ and $\lambda = \frac{\triangle t}{\triangle x} = \frac{2}{3}$. The boundary conditions are as in [2].

Plot the results at $t = 1$, together with the exact solution. Comment and compare the obtained results. Note that here the FTBS scheme is the upwind scheme, in the case $a = 1 > 0$ (positive speed of propagation).

[4] Consider the nonlinear equation $u_t + u_x = \cos^2 u$, approximated by the Lax-Wendroff scheme with $R_{k,h} f_m^n = f_m^n$, treating the $\cos^2 u$ term as $f(t,x)$. Show that the obtained scheme is first order accurate (use $\lambda = \frac{k}{h}$).

[5] Consider the box scheme

$$\begin{aligned}
&\frac{1}{2k}[(v_m^{n+1} + v_{m+1}^{n+1}) - (v_m^n + v_{m+1}^n)] \\
+&\frac{a}{2h}[(v_{m+1}^{n+1} - v_m^{n+1}) + (v_{m+1}^n - v_m^n)] \\
&= \frac{1}{4}(f_{m+1}^{n+1} + f_m^{n+1} + f_{m+1}^n + f_m^n)
\end{aligned}$$

(a) Show that the scheme is an approximation to the one-way wave equation $u_t + au_x = f$ that is accurate of order (2,2).

(b) Show that the scheme is stable for all values of λ.