Math 269B Homework #4 Due date: Friday, February 17
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[1] Consider the one-way wave equation

\[ \frac{\partial u}{\partial t} + \frac{\partial u}{\partial x} = 0. \]

Analyze the stability and the order of accuracy of the following angled derivative method

\[ u^{n+2}_j = (1 - 2\lambda)(u^{n+1}_j - u^{n+1}_{j-1}) + u^n_{j-1}, \quad n \geq 0, \]

with \( \lambda = \frac{\Delta t}{\Delta x}. \)

[2] **Computational assignment.** Consider the one-way wave equation

\[ \frac{\partial u}{\partial t} + \frac{\partial u}{\partial x} = 0, \quad t > 0, \quad 0 < x < 2; \]

\[ u(x, 0) = e^{-100(x-1/2)^2} \sin(20\pi x), \]

\[ u(0, t) = u(2, t) = 0, \quad t \geq 0. \]

Find approximations of the exact solution at \( t = 1, \) using \( \Delta x = \frac{1}{80} \) and \( \lambda = \frac{\Delta t}{\Delta x} = \frac{2}{3}, \) by three methods: the leapfrog method, the Lax-Wendroff method, and the angled derivative method.

Comment and compare the obtained approximations, function of accuracy, dissipation, dispersion, conservation (a non-dissipative scheme is called conservative). Give and plot the exact solution also at \( t = 1. \)

[3] **Computational assignment.** Consider now the same equation as in [2], but with the initial condition

\[ u(x, 0) = \begin{cases} 
1, & \text{if } \frac{1}{4} \leq x < \frac{3}{4}, \\
0, & \text{otherwise}. 
\end{cases} \]

Find numerical approximations using four methods: the three methods used in [2] and the forward-in-time backward-in-space method (FTBS). In all cases, take \( \Delta x = \frac{1}{100} \) and \( \lambda = \frac{\Delta t}{\Delta x} = \frac{2}{3}. \) The boundary conditions are as in [2].

Plot the results at \( t = 1, \) together with the exact solution. Comment and compare the obtained results. Note that here the FTBS scheme is the upwind scheme, in the case \( a = 1 > 0 \) (positive speed of propagation).
Consider the nonlinear equation $u_t + u_x = \cos^2 u$, approximated by the Lax-Wendroff scheme with $R_{k,h}f^n_m = f^n_m$, treating the $\cos^2 u$ term as $f(t,x)$. Show that the obtained scheme is first order accurate (use $\lambda = \frac{k}{h}$).

Consider the box scheme

\[
\frac{1}{2k}[(v_{m+1}^{n+1} + v_{m+1}^n) - (v_m^n + v_{m+1}^n)] \\
+ \frac{a}{2h}[(v_{m+1}^{n+1} - v_m^{n+1}) + (v_{m+1}^n - v_m^n)] \\
= \frac{1}{4}(f_{m+1}^{n+1} + f_{m+1}^n + f_{m+1}^n + f_m^n)
\]

(a) Show that the scheme is an approximation to the one-way wave equation $u_t + au_x = f$ that is accurate of order (2,2).

(b) Show that the scheme is stable for all values of $\lambda$. 

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