Math 269B. Homework #3. Due date: Friday Feb. 3 or Monday Feb. 6. (there are 4 exercises)

[**1**] (*computational assignment*) Solve the one-way wave equation with variable coefficients

$$u_t + (1+\alpha x)u_x = 0$$

on the interval $[-3,3]$ and $0 \le t \le 2$ with the Lax-Friedrichs scheme

$$v_m^{n+1} = \frac{1}{2}(v_{m+1}^n + v_{m-1}^n) - \frac{1}{2}a(t_n, x_m)\lambda(v_{m+1}^n - v_{m-1}^n),$$

where $a(t,x) = (1+\alpha x)$. Consider $\alpha = -0.5$, $\lambda = 1$. Demonstrate that the instability phenomena occur where $|(1+\alpha x_m)| > 1$. Use the initial data

$$u_0(x) = \begin{cases} 1-|x| & \text{if } |x| \le 1, \\ 0 & \text{otherwise.} \end{cases}$$

Specify the solution to be 0 at both boundaries. You can use a grid spacing of 0.1 ($h = 0.1$).

[**2**] Show that the scheme

$$\frac{v_{l,m}^{n+1} - \frac{1}{4}(v_{l+1,m+1}^n + v_{l-1,m+1}^n + v_{l+1,m-1}^n + v_{l-1,m-1}^n)}{k} + a\triangle_0^x v_{l,m}^n + b\triangle_0^y v_{l,m}^n = 0$$

for the equation $u_t + au_x + bu_y = 0$, with $\triangle x = \triangle y = h$, is stable if and only if $(|a|+|b|)\lambda \le 1$.

[**3**] Consider the system

$$\begin{cases} u_t = av_x \\ v_t = au_x. \end{cases}$$

(a) Give a single PDE equivalent with this system.

(b) Apply the Lax-Friedrichs scheme to the system. Is the obtained system of finite differences consistent with the system of PDE's ?

(c) Apply von Neumann stability analysis to the vector scheme, in the case when $\lambda = \frac{k}{h} = constant$, substituting

$$\begin{pmatrix} u_m^n \\ v_m^n \end{pmatrix} = g^n e^{im\theta} \begin{pmatrix} u^0 \\ v^0 \end{pmatrix}$$

in the difference vector equation, where $\begin{pmatrix} u^0 \\ v^0 \end{pmatrix}$ is a constant vector both in space and in time.

[**4**] Consider the system of equations

$$\begin{pmatrix} u \\ v \end{pmatrix}_t = A \begin{pmatrix} u \\ v \end{pmatrix}_x ; \qquad A = \begin{pmatrix} 1 & \alpha \\ 0 & 1 \end{pmatrix},$$

which is approximated by the Lax-Friedrichs scheme,

$$V_j^{n+1} = \frac{1}{2}(V_{j+1}^n + V_{j-1}^n) + \frac{\triangle t}{2\triangle x} A (V_{j+1}^n - V_{j-1}^n),$$

where $V_j^n \approx (u(x_j, t_n), v(x_j, t_n))$.

Determine the stability properties depending on whether $\alpha \neq 0$ or $\alpha = 0$.

Hint: If $\alpha = 0$ this becomes an uncoupled system of two scalar one-way wave equations, and the stability condition of the scalar Lax-Friedrichs scheme applies for each equation. If $\alpha \neq 0$, show that the scheme is unstable (you can compute the amplification matrix $G(\theta)$, for constant λ, and show that the matrices G^n are unbounded in some matrix norm, for at least one value of θ). Compute an explicit form of G^n.