Math 269B, Hw #1
Due on: Friday, January 20

[1] (computational assignment) For \( x \in [-1, 3] \) and \( t \in [0, 2.4] \), solve the one-way wave equation

\[
    u_t + u_x = 0,
\]

with the initial data

\[
u(0, x) = \begin{cases}
    \cos^2 \pi x & \text{if } |x| \leq \frac{1}{2}, \\
    0 & \text{otherwise},
\end{cases}
\]

and the boundary data \( u(t, -1) = 0 \).

Use the Lax-Friedrichs scheme with \( \lambda = 0.8 \) and \( \lambda = 1.6 \) and \( h = 1/10 \), \( h = 1/20 \). At the right boundary use the condition \( v_{M-1} = v_{M-1}^{n+1} \), where \( x_M = 3 \).

The Lax-Friedrichs scheme for \( u_t + au_x = 0 \) is

\[
    \frac{v_{m+1}^{n+1} - \frac{1}{2}(v_{m+1}^n + v_{m-1}^n)}{k} + a\frac{v_{m+1}^n - v_{m-1}^n}{2h} = 0.
\]

Graph or plot solutions at the last time they were computed. What can you say about the behavior of the numerical solutions as a function of \( h \) or \( \lambda \)?

You should turn in the code in the language of your choice, the plots and your interpretation of the tests performed.

[2] Show that the forward-time central-space scheme is consistent with the equation \( u_t + au_x = 0 \).

[3] Show that the following scheme is consistent with the equation

\[
    u_t + cu_{tx} + au_x = f
\]

\[
    \frac{v_{m+1}^n - v_m^n}{k} + c\frac{v_{m+1}^n - v_{m-1}^n - v_{m+1}^n + v_{m+1}^n}{2kh} + a\frac{v_{m+1}^n - v_{m-1}^n}{2h} = f^n.
\]

[4] Show that schemes of the form

\[
    v_{m+1}^n = \alpha v_{m+1}^n + \beta v_{m-1}^n
\]

are stable if \(|\alpha| + |\beta| \leq 1\). Conclude that the Lax-Friedrichs scheme is stable if \(|a\lambda| \leq 1\).