269B, Winter 2005

Final exam: Wednesday, March 16, 2003, 2:00pm, MS 5138.

• Office hours on Tuesday, March 15: 2-4pm, MS 7620-D, or by appointment.

If needed, a review session may be added sometime on Tuesday afternoon, time and place will be decided in class.

All topics discussed are covered for the final.

Practice problems for the final

(do not work only these problems; more exercises can be obtained from the past numerical analysis quals).

[**1**] Consider the convection-diffusion equation

$$u_t + a u_x = u_{xx},$$

to be solved for $t > 0$, $u(x,0)$ given and $u(x,t)$ periodic in x, with the constant $a > 0$.

(a) Is this a well-posed initial value problem ? Explain.

(b) Consider the difference approximation

$$\frac{u_i^{n+1} - u_i^n}{\triangle t} + a\frac{u_{i+1}^n - u_{i-1}^n}{2\triangle x} = \frac{u_{i+1}^n - 2u_i^n + u_{i-1}^n}{(\triangle x)^2}.$$

For which values of $\triangle t$, $\triangle x$, a do we have a scheme which satisfies a maximum norm stability as $\triangle x \to 0$? Are you satisfied with this result ? Explain.

(c) Set up a scheme which is explicit, consistent, and satisfies the maximum norm stability for

$$a\frac{\triangle t}{\triangle x} + 2\frac{\triangle t}{(\triangle x)^2} \le 1.$$

Explain.

[**2**] To solve

$$u_t + a u_x = 0 \ \text{ for } t > 0, \ 0 \le x \le 1,$$

$u(x,0) = \phi(x)$ smooth, u periodic in x, $u(x+1,t) = u(x,t)$, we use:

$$\frac{1}{2\triangle t}\Big[(v_j^{n+1} + v_{j+1}^{n+1}) - (v_j^n + v_{j+1}^n)\Big] + \frac{a}{2\triangle x}\Big[v_{j+1}^{n+1} - v_j^{n+1} + v_{j+1}^n - v_j^n\Big] = 0.$$

For what values of $\frac{\triangle t}{\triangle x}$, if any, does this converge ? At what rate ? Explain your answers.

(Recall that for smooth initial data, the order of accuracy of the scheme gives the order of accuracy of the solution.)

(Note that sometimes the computation of $|g(\theta)|^2$ is simplified if you use $e^{i\theta} = (e^{i\theta/2})^2$, $e^{-i\theta} = (e^{-i\theta/2})^2$, and $1 = e^{i\theta/2}e^{-i\theta/2}$. These could have been used for [1] from assignment #7 (the angled derivative method).)

[3] Consider the differential equation

$$u_t = u_{xx} + u_{yy} + bu_{xy} \quad \text{for } t > 0,\ 0 < x < 1,\ 0 < y < 1,$$

with $u = 0$ on the boundary, and $u(0, x, y) = \phi(x, y)$, a smooth function.

(a) For what values of b can you obtain a convergent, unconditionally stable finite difference scheme ?

(b) Construct such a scheme. Explain your answers.

Note that this is a particular case of the more general problem, given next.

[3'] Consider the PDE

$$u_t = Au_{xx} + 2Bu_{xy} + Cu_{yy},$$

with associated smooth initial and boundary conditions.

(a) For what constants A, B, C is this a well-posed parabolic equation ? (answer: if $A > 0,\ C > 0,\ B^2 < AC$).

(b) Apply the Θ-method to this problem (as outlined below).

Let $\Phi^n_{j,l} = \frac{A}{(\triangle x)^2}(u^n_{j+1,l} - 2u^n_{j,l} + u^n_{j-1,l}) + \frac{B}{2\triangle x \triangle y}(u^n_{j+1,l+1} - u^n_{j-1,l+1} - u^n_{j+1,l-1} + u^n_{j-1,l-1}) + \frac{C}{(\triangle y)^2}(u^n_{j,l+1} - 2u^n_{j,l} + u^n_{j,l-1})$.

The general Θ-method scheme, for $0 \le \Theta \le 1$, is

$$\frac{u^{n+1}_{j,l} - u^n_{j,l}}{\triangle t} = \Theta\Phi^{n+1}_{j,l} + (1-\Theta)\Phi^n_{j,l}.$$

(c) Obtain the stability conditions of the scheme, function of Θ.

Hint:

- Use $u^n_{j,l} = g^n e^{ij\theta_1} e^{il\theta_2}$ as usual.

- Compute the amplification factor g of the scheme, and write it in the form $g = \frac{1+(1-\Theta)\Psi}{1-\Theta\Psi}$, for $\Psi = \Psi(\theta_1, \theta_2)$ to be determined.

- Notice that Ψ is a real analytic periodic function of (θ_1, θ_2), therefore $g(\theta_1, \theta_2)$ is also real.

- To see under what conditions $|g(\theta_1, \theta_2)| \leq 1$, find first the minimum and maximum values of Ψ (to find the extema of Ψ, write a linear system of two equations obtained from $\frac{\partial \Psi}{\partial \theta_1} = \frac{\partial \Psi}{\partial \theta_2} = 0$; consider the unknowns $\sin\theta_1$ and $\sin\theta_2$).

- Using the lower and upper bounds of Ψ, you will obtain the stability conditions of the scheme (function of Θ).

[4] Consider constructing a numerical method to solve $u_t = u_{xx}$ for $t > 0$, $0 \leq x \leq 1$, with periodic boundary conditions:

$$u(0, t) = u(1,)$$

and smooth initial data

$$u(x, 0) = \phi(x).$$

Would you rather use the approximation (A) or (B):

$$\frac{u_i^{n+1} - u_i^{n-1}}{2\triangle t} = \frac{u_{i+1}^n - 2u_i^n + u_{i-1}^n}{(\triangle x)^2} \qquad (A)$$

or

$$\frac{u_i^{n+1} - u_i^{n-1}}{2\triangle t} = \frac{u_{i+1}^n - (u_i^{n+1} + u_i^{n-1}) + u_{i-1}^n}{(\triangle x)^2} \qquad (B)$$

Describe the stability and convergence properties of both methods.

[5] Consider the differential equation $u_t = b(u_{xx} + u_{yy})$, with the constant $b > 0$. Apply the F-T, C-S scheme to this problem. Give the first terms of the truncation error, and show that the scheme is stable if $\nu_x + \nu_y \leq \frac{1}{2}$, where $\nu_x = \frac{b\triangle t}{(\triangle x)^2}$, $\nu_y = \frac{b\triangle t}{(\triangle y)^2}$.

[6] The linearised 1-D forms of the isentropic compressible fluid flow equations are

$$\rho_t + q\rho_x + w_x = 0,$$
$$w_t + qw_x + a^2\rho_x = 0,$$

where a and q are positive constants. Show that an explicit scheme which uses central differences for the x-derivatives is always unstable. By adding the extra terms arising from a Lax-Wendroff scheme, derive a conditionally stable scheme and find the stability condition.

[7] Consider the second order equation

$$u_{tt} + 2bu_{tx} - a^2 u_{xx} - cu_x - du_t = 0,$$

to be solved for $t > 0$, periodic in x, of period 1.

(a) Write it as an equivalent first order system.

Hint: you can use the substitutions $u^1 = u_x$ and $u^2 = u_t$.

(b) For which values of the real numbers a, b is the corresponding initial value problem well-posed ?

(c) Set up a convergent finite difference approximation for the well-posed initial value problem.

Justify your answers.