269B, Winter 2005, Vese
HW #8, due on Monday, March 14 or Wednesday, March 16
(no late homework accepted)

REMINDER:
FINAL EXAM on Wednesday, March 16, 2-5pm, MS 5138.

[**1**] Consider the system of partial differential equations

$$(1) \qquad \begin{cases} u_t + u_x + v_x = 0, & u(x,0) = u_0(x), \\ v_t + u_x - v_x = 0, & v(x,0) = v_0(x). \end{cases}$$

(a) Write the system in matrix form $U_t + AU_x = \vec{0}$, with $U(x,t) = \begin{pmatrix} u(x,t) \\ v(x,t) \end{pmatrix}$, and find the matrix A.

(b) Is this a hyperbolic system ? Explain.

(c) Reduce, if possible, the system (1) to a set of independent scalar hyperbolic equations.

(d) Obtain the exact solution of the system (1).

(e) Let $U_j^n \approx \begin{pmatrix} u(x_j,t_n) \\ v(x_j,t_n) \end{pmatrix}$. Consider the FTCS scheme applied to the system (1) in matrix form:

$$(2) \qquad \frac{1}{\triangle t}(U_j^{n+1} - U_j^n) + \frac{1}{2h}A(U_{j+1}^n - U_{j-1}^n) = \vec{0}.$$

Find the nonsingular amplification matrix G for this scheme by substituting $U_j^n \approx G^n e^{ij\theta}$ into the difference equation (2).

(f) Analyze the stability of the scheme, using the amplification matrix G (Hint: note that if $G = P(A)$, with P a polynomial or a rational function, then the eigenvalues g_i of G are given by $g_i = P(a_i)$, where a_i are the eigenvalues of A, and both matrices are diagonalizable by the same matrix).

[2] Consider the scalar hyperbolic equation $u_t + au_x + bu_y = 0$ in two space dimensions. Consider the discretizations $(x_l, y_m, t^n) = (l\triangle x, m\triangle y, n\triangle t)$, and $u_{l,m}^n \approx u(x_l, y_m, t^n)$.

Show that the scheme

$$\frac{u_{l,m}^{n+1} - \frac{1}{4}(u_{l+1,m+1}^n + u_{l-1,m+1}^n + u_{l+1,m-1}^n + u_{l-1,m-1}^n)}{k} + a\triangle_0^x u_{l,m}^n + b\triangle_0^y u_{l,m}^n = 0$$

for the equation $u_t + au_x + bu_y = 0$, with $\triangle x = \triangle y = h$, is stable if and only if $(|a| + |b|)\lambda \leq 1$.

Here $\triangle_0^x$, $\triangle_0^y$ are the usual second order central approximations to u_x and u_y. (HINT: substitute in the difference scheme $u^n_{l,m}$ by $g^n e^{il\theta_1} e^{im\theta_2}$, where $g = g(\theta_1, \theta_2)$ if $\lambda = \triangle t/h$ is constant, with $\theta_1 = h\xi_1$ and $\theta_2 = h\xi_2$, (ξ_1, ξ_2) being the variables in the frequency domain).

[3]

(a) Consider the diffusion equation in one space dimension $u_t = bu_{xx} + f(x,t)$, $b > 0$. Show that the Du Fort-Frankel scheme, which is a modification of the leapfrog scheme,

$$\frac{u_j^{n+1} - u_j^{n-1}}{2\triangle t} = b\frac{u^n_{j+1} - (u_j^{n+1} + u_j^{n-1}) + u^n_{j-1}}{h^2} + f_j^n$$

has the order of accuracy given by $O(h^2) + O(\triangle t^2) + O(\frac{\triangle t^2}{h^2})$, and also that the scheme is unconditionally stable.

(b) Show that the two-dimensional Du Fort-Frankel scheme for the equation $u_t = b(u_{xx} + u_{yy}) + f(x, y, t)$ given by

$$\frac{u^{n+1}_{l,m} - u^{n-1}_{l,m}}{2\triangle t} = b\frac{u^n_{l+1,m} + u^n_{l-1,m} + u^n_{l,m+1} + u^n_{l,m-1} - 2(u^{n+1}_{l,m} + u^{n-1}_{l,m})}{h^2} + f^n_{l,m},$$

where $\triangle x = \triangle y = h$, is unconditionally stable.

[4] Consider the equation $u_t + a(x,t)u_x = 0$, where $a(x,t)$ may change sign. We know that when $a = constant$, if $a > 0$ the initial profile propagates to the right, while if $a < 0$ it propagates to the left. Therefore, if $a > 0$ the information from the left is propagated to the right, and vice-versa. Moreover, if $a > 0$, then the FTFS scheme is unstable, while if $a < 0$, then the FTBS scheme is unstable.

Based on these remarks, propose an "upwind" scheme for $u_t + a(x,t)u_x = 0$, of first order, that combines FTFS and FTBS. Give a sufficient condition of stability for the scheme with variable coefficients.