269B, Winter 2005, Vese

HW #7, due on Friday, March 4

[1] Consider the one-way wave equation

$$\frac{\partial u}{\partial t} + \frac{\partial u}{\partial x} = 0.$$

Analyze the stability and the order of accuracy of the following *angled derivative method*

$$u_j^{n+2} = (1-2\lambda)(u_j^{n+1} - u_{j-1}^{n+1}) + u_{j-1}^n, \quad n \geq 0,$$

with $\lambda = \frac{\triangle t}{\triangle x}$.

[2] Consider the system

$$\begin{cases} u_t = av_x \\ v_t = au_x. \end{cases}$$

(a) Give a single PDE equivalent with this system.

(b) Apply the Lax-Friedrichs scheme to the system. Is the obtained system of finite differences consistent with the system of PDE's ?

(c) Apply von Neumann stability analysis to the vector scheme, in the case when $\lambda = \frac{\triangle t}{h} = \mathit{constant}$, substituting

$$\begin{pmatrix} u_j^n \\ v_j^n \end{pmatrix} = g^n e^{ij\theta} \begin{pmatrix} u^0 \\ v^0 \end{pmatrix}$$

in the difference vector equation, where $\begin{pmatrix} u^0 \\ v^0 \end{pmatrix}$ is a constant vector both in space and in time.

[3] Consider the nonlinear equation $u_t + u_x = \cos^2 u$, approximated by the Lax-Wendroff scheme with $R_{k,h} f_m^n = f_m^n$, treating the $\cos^2 u$ term as $f(x,t)$. Show that the obtained scheme is first order accurate (use $\lambda = \frac{\triangle t}{h}$).

[4] *Computational assignment.* Consider the one-way wave equation

$$\frac{\partial u}{\partial t} + \frac{\partial u}{\partial x} = 0, \quad t > 0, \quad 0 < x < 2,$$
$$u(x,0) = e^{-100(x-1/2)^2} \sin(20\pi x),$$
$$u(0,t) = u(2,t) = 0, \quad t \geq 0.$$

Find approximations of the exact solution at $t = 1$, using $\triangle x = \frac{1}{80}$ and $\lambda = \frac{\triangle t}{\triangle x} = \frac{2}{3}$, by three methods: the leapfrog method (C-T, C-S), the Lax-Wendroff method, and the angled derivative method.

Comment and compare the obtained approximations. You can initialize the leapfrog scheme at level u_j^1 by the exact solution, or by an approximation obtained using a one-step method.

Give and plot the exact solution at $t = 1$.