269B, Winter 2005, Vese
HW #6, due on Friday, Febryary 25

[1] Consider the system

$$\begin{cases} u_t + 2u_x + v_x = 0 \\ v_t + u_x + 2v_x = 0 \end{cases},$$

with initial data $u_0(x) = u(x,0) = \begin{cases} 1 \text{ if } |x| \leq 1 \\ 0 \text{ if } |x| > 1 \end{cases}$, $v(x,0) = v_0(x) = 0$.

(a) Write the system in matrix-vector form.

(b) Find the exact solution of the system. Hint: transform the system into a set of two uncoupled one-way wave equations.

[2] Consider the one-way wave equation $u_t + au_x = 0$ for $t > 0$ and $x \in R$, with the initial condition $u(x,0) = u_0(x)$.

(a) Give the exact solution of the equation.

(b) Show that the leapfrog scheme (C-T,C-S) is consistent with the one-way wave equation $u_t + au_x = 0$. Find the order of the scheme.

(c) Show that the leapfrog scheme with $\lambda = \frac{\triangle t}{h} = 1$ applied to $u_t + u_x = 0$ gives the exact solution (let u_j^0 and u_j^1 be given by the exact solution at $t = 0$ and $t = \triangle t$, for all $j \in Z$).

[3] Consider the modified Lax-Friedrichs scheme for the one-way wave equation $u_t + au_x = f(x,t)$,

$$u_j^{n+1} = \frac{1}{2}(u_{j+1}^n + u_{j-1}^n) - \frac{a\lambda}{1 + (a\lambda)^2}(u_{j+1}^n - u_{j-1}^n) + \triangle t f_j^n.$$

(a) Analize the consistency of the scheme.

(b) Show that this explict scheme is stable for all values of λ. Discuss the relation of this explicit and unconditionally stable scheme with the Courant-Friedrichs-Lewy Theorem.

CFL Thm: There are no explicit, unconditionally stable, consistent finite difference schemes for hyperbolic systems of partial differential equations.

[4] Show that the scheme forward-time central-space for $u_t + au_x = bu_{xx}$ satisfies the condition $|g| \leq 1$ if and only if $\triangle t \leq 2b/a^2$.