Math 269B, Hw #5
Due date: Monday, February 14, 2005

[1] (*computational assignment*) Solve the one-way wave equation with variable coefficients

$$u_t + (1+\alpha x)u_x = 0$$

on the interval $[-3,3]$ and $0 \le t \le 2$ with the Lax-Friedrichs scheme

$$u_j^{n+1} = \frac{1}{2}(u_{j+1}^n + u_{j-1}^n) - \frac{1}{2}a(x_j,t_n)\lambda(u_{j+1}^n - u_{j-1}^n),$$

where $a(x,t) = (1+\alpha x)$. Consider $\alpha = -0.5$, $\lambda = 1$. Demonstrate that the instability phenomena occur where $|(1+\alpha x_j)| > 1$. Use the initial data

$$u_0(x) = \begin{cases} 1-|x| & \text{if } |x| \le 1, \\ 0 & \text{otherwise.} \end{cases}$$

Specify the solution to be 0 at both boundaries. You can use a grid spacing of 0.1 ($h = 0.1$).

[2] Show that the scheme

$$\frac{u_j^{n+1} - u_j^n}{\triangle t} + a\frac{u_{j+2}^n - 3u_{j+1}^n + 3u_j^n - u_{j-1}^n}{h^3} = f_j^n$$

is consistent with the equation $u_t + au_{xxx} = f$ and, if $\nu = \frac{\triangle t}{h^3}$ is constant, then it is stable when $0 \le a\nu \le \frac{1}{4}$ (note that for stability, it is sufficient to consider the homogeneous equation).

[3] Consider the box scheme

$$\begin{aligned} &\frac{1}{2\triangle t}[(u_j^{n+1} + u_{j+1}^{n+1}) - (u_j^n + u_{j+1}^n)] \\ &+\frac{a}{2h}[(u_{j+1}^{n+1} - u_j^{n+1}) + (u_{j+1}^n - u_j^n)] \\ &\quad = \frac{1}{4}(f_{j+1}^{n+1} + f_j^{n+1} + f_{j+1}^n + f_j^n) \end{aligned}$$

(a) Show that the scheme is an approximation to the one-way wave equation $u_t + au_x = f$ that is accurate of order (2,2).
(b) Show that the scheme is stable for all values of $\lambda = \frac{\triangle t}{h}$.