Math 269B, Hw #4
Due date: Monday, February 7, 2005

[1] (computational assignment) For \( x \in [-1, 3] \) and \( t \in [0, 2.4] \), solve the one-way wave equation

\[
 u_t + u_x = 0,
\]

with the initial data

\[
 u(0, x) = \begin{cases} 
 \cos^2 \pi x & \text{if } |x| \leq \frac{1}{2}, \\
 0 & \text{otherwise},
\end{cases}
\]

and the boundary data \( u(t, -1) = 0 \).

Use the Lax-Friedrichs scheme with \( \lambda = 0.8 \) and \( \lambda = 1.6 \) and \( h = 1/10 \), \( h = 1/20 \), where \( \lambda = \frac{\Delta t}{h} \). At the right boundary use the condition \( v_{M+1}^{n+1} = v_{M-1}^n \), where \( x_M = 3 \).

The Lax-Friedrichs scheme for \( u_t + au_x = 0 \) is

\[
 \frac{u_j^{n+1} - \frac{1}{2}(u_{j+1}^n + u_{j-1}^n)}{\Delta t} + a \frac{u_{j+1}^n - u_{j-1}^n}{2h} = 0.
\]

Graph or plot solutions at the last time they were computed. What can you say about the behavior of the numerical solutions as a function of \( h \) or \( \lambda \)?

You should turn in the code in the language of your choice, the plots and your interpretation of the tests performed.

[2] Show that schemes of the form

\[
 u_j^{n+1} = \alpha u_{j+1}^n + \beta u_{j-1}^n
\]

are stable if \( |\alpha| + |\beta| \leq 1 \). Conclude that the Lax-Friedrichs scheme is stable if \( |\alpha \lambda| \leq 1 \).

[3] Show that schemes of the form

\[
 \alpha u_{j+1}^{n+1} + \beta u_{j-1}^{n+1} = u_j^n
\]

are stable if \( ||\alpha| - |\beta|| \geq 1 \). Conclude that the reverse Lax-Friedrichs scheme,

\[
 \frac{1}{2}(u_{j+1}^{n+1} + u_{j-1}^{n+1}) - u_j^n + a \frac{u_{j+1}^{n+1} - u_{j-1}^{n+1}}{2h} = 0
\]

is stable if \( |\alpha \lambda| \geq 1 \).
[4] Show that the following scheme is consistent with the equation
\[ u_t + cu_{tx} + au_x = f \]
\[
\frac{u_j^{n+1} - u_j^n}{\Delta t} + c \frac{u_{j+1}^{n+1} - u_{j-1}^{n+1} - u_{j+1}^n + u_{j-1}^n}{2\Delta th} + a \frac{u_{j+1}^n - u_{j-1}^n}{2h} = f_j^n.
\]