269B Winter 2005, Vese

NOTE: Monday, January 24, from 2-3pm, the T.A. will hold a D.S. instead of the regular lecture. The other class meetings on Wednesday, Thursday and Friday will be as usual.

Homework #3

Due on Friday, January 28,2005

[**1**] If $v = (v_m)$ is a grid function defined for all integers m, with spacing between the grid points h, let's define the discrete Fourier transform by

$$\hat{v}(\xi) = \frac{1}{\sqrt{2\pi}} \sum_{m=-\infty}^{\infty} e^{-imh\xi} v_m h$$

for $\xi \in [-\pi/h, \pi/h]$, and then the inversion formula is

$$v_m = \frac{1}{\sqrt{2\pi}} \int_{-\pi/h}^{\pi/h} e^{imh\xi} \hat{v}(\xi) d\xi.$$

Introduce the definitions

$$\|\hat{v}\|_h^2 := \int_{-\pi/h}^{\pi/h} |\hat{v}(\xi)|^2 d\xi, \quad \|v\|_h^2 := \sum_{m=-\infty}^{\infty} |v_m|^2 h.$$

Prove the discrete Parserval's relation

$$\|\hat{v}\|_h^2 = \|v\|_h^2.$$

Hint: Begin with $\|\hat{v}\|_h^2$, write $|\hat{v}(\xi)|^2 = \overline{\hat{v}(\xi)}\hat{v}(\xi)$, and express $\hat{v}(\xi)$ using the definition above. Assume that we can interchange the integration and summation operations.

[**2**] Evaluate the integral

$$\frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} e^{i\omega(x-y)} e^{-b\omega^2 t} d\omega,$$

by considering the function

$$F(\alpha) = \frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} e^{i\omega\alpha} e^{-\omega^2} d\omega.$$

Hint: Show that $F(0) = 1$ and that $F'(\alpha) = -\frac{1}{2}\alpha F(\alpha)$.

[3] Consider the one-dimensional heat equation $u_t = bu_{xx}$, $b > 0$, $t \geq 0$, and $u(x,0) = u_0(x)$.

(a) As in class, show that

$$u(x,t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{i\omega x} e^{-b\omega^2 t} \hat{u}_0(\omega) d\omega.$$

(b) Using Parseval's relation $\|v\|_{L^2}^2 = \int_{-\infty}^{\infty} |v(x)|^2 dx = \int_{-\infty}^{\infty} |\hat{v}(\omega)|^2 d\omega = \|\hat{v}\|_{L^2}^2$, show that

$$\|u(\cdot,t)\|_{L^2}^2 \leq \|u_0(\cdot)\|_{L^2}^2,$$

for any $t \geq 0$.

(c) Using the definition of $\hat{u}_0$ in (a), problem [2], and interchanging the order of integration, show that

$$u(x,t) = \frac{1}{\sqrt{4\pi bt}} \int_{-\infty}^{\infty} e^{-(x-y)^2/4bt} u_0(y) dy.$$