269B Winter 2005, Vese
Homework #2
Due on Friday, January 21, 2005

[**1**] Consider the convection-diffusion equation $u_t + au_x = bu_{xx}$, with $b > 0$, and a, b are constant parameters. Let $y = x - at$ and set $w(y,t) = u(y+at, t)$. Find an equation satisfied by $w(y,t)$.

[**2**] Show that a scheme for $u_t = bu_{xx}$, $b > 0$, of the form

$$u_j^{n+1} = \alpha u_j^n + \frac{1-\alpha}{2}(u_{j+1}^n + u_{j-1}^n),$$

with α constant as $\triangle x$ and $\triangle t$ tend to zero, is consistent with the heat equation only if $\alpha = 1 - 2b\nu$.

[**3**] Determine the order of the scheme BTCS for the heat equation $u_t = bu_{xx}$,

$$\frac{u_j^{n+1} - u_j^n}{\triangle t} = b\frac{u_{j+1}^{n+1} - 2u_j^{n+1} + u_{j-1}^{n+1}}{h^2}.$$

[**4**] Consider the Crank-Nicolson scheme for $u_t = bu_{xx}$,

$$\frac{u_j^{n+1} - u_j^n}{\triangle t} = \frac{1}{2}b\frac{u_{j+1}^{n+1} - 2u_j^{n+1} + u_{j-1}^{n+1}}{h^2} + \frac{1}{2}b\frac{u_{j+1}^n - 2u_j^n + u_{j-1}^n}{h^2}.$$

(a) Show that this scheme satisfies the *maximum norm stability*

$$\|u^{n+1}\|_\infty \leq \|u^n\|_\infty$$

for all solutions if $b\nu \leq 1$. *Hint:* Show that if $u_{j'}^{n+1}$ is the largest value of u_j^{n+1}, then

$$u_{j'}^{n+1} \leq -\frac{b\nu}{2}u_{j'-1}^{n+1} + (1+b\nu)u_{j'}^{n+1} - \frac{b\nu}{2}u_{j'+1}^{n+1} \leq \|u^n\|_\infty.$$

(b) Find the order of the scheme.