Final exam: Monday, March 17, 2003, 3:00pm-6:00pm, MS 5127.

• Office hours on Saturday, March 15: 2-4pm, MS 7620-D.

• Review session on Friday, 4pm (we will solve some of the practice problems). Room 5117.

• The topic of elliptic (stationary) equations, discussed in Chapter 12, and in the last lecture on Friday, will not be covered for the final exam.

Practice problems for the final

(these are just additional sample problems, given at past Num. Anal. qualifying exams; but you have to review all the material for the final exam)

• You have to be able (in some cases) to give an example of a 1st order or second order scheme, or of an unconditionally (or just conditionally) stable scheme, or an explicit or implicit scheme. These are among the main examples of schemes discussed in class.

[**1**] Consider the convection-diffusion equation

$$u_t + a u_x = u_{xx},$$

to be solved for $t > 0$, $u(0,x)$ given and $u(t,x)$ periodic in x, with the constant $a > 0$.

(a) Is this a well-posed initial value problem ? Explain.

(b) Consider the difference approximation

$$\frac{u_i^{n+1} - u_i^n}{\triangle t} + a\frac{u_{i+1}^n - u_{i-1}^n}{2\triangle x} = \frac{u_{i+1}^n - 2u_i^n + u_{i-1}^n}{(\triangle x)^2}.$$

For which values of $\triangle t$, $\triangle x$, a do we have a scheme which satisfies a maximum norm stability as $\triangle x \to 0$? Are you satisfied with this result ? Explain.

(c) Set up a scheme which is explicit, consistent, and satisfies the maximum norm stability for

$$a\frac{\triangle t}{\triangle x} + 2\frac{\triangle t}{(\triangle x)^2} \le 1.$$

Explain.

[2] Consider the second order equation

$$u_{tt} + 2bu_{tx} - a^2 u_{xx} - cu_x - du_t = 0,$$

to be solved for $t > 0$, periodic in x, of period 1.

(a) Write it as an equivalent first order system.

(b) For which values of the real numbers a, b is the corresponding initial value problem well-posed ?

(c) Set up a convergent finite difference approximation for the well-posed initial value problem.

Justify your answers.

[3] To solve

$$u_t + au_x = 0 \quad \text{for } t > 0,\ 0 \le x \le 1,$$

$u(0, x) = \phi(x)$ smooth, u periodic in x, $u(t, x+1) = u(t, x)$, we use:

$$\frac{1}{2\triangle t}\Big[(v_j^{n+1} + v_{j+1}^{n+1}) - (v_j^n + v_{j+1}^n)\Big] + \frac{a}{2\triangle x}\Big[v_{j+1}^{n+1} - v_j^{n+1} + v_{j+1}^n - v_j^n\Big] = 0.$$

For what values of $\frac{\triangle t}{\triangle x}$, if any, does this converge ? At what rate ? Explain your answers.

[4] Consider the differential equation

$$u_t = u_{xx} + u_{yy} + bu_{xy} \quad \text{for } t > 0,\ 0 < x < 1,\ 0 < y < 1,$$

with $u = 0$ on the boundary, and $u(0, x, y) = \phi(x, y)$, a smooth function.

(a) For what values of b can you obtain a convergent, unconditionally stable finite difference scheme ?

(b) Construct such a scheme. Explain your answers.

[5] Consider constructing a numerical method to solve $u_t = u_{xx}$ for $t > 0$, $0 \le x \le 1$, with periodic boundary conditions:

$$u(t, 0) = u(t, 1)$$

and smooth initial data

$$u(0, x) = \phi(x).$$

Would you rather use the approximation (A) or (B):

$$\frac{u_i^{n+1} - u_i^{n-1}}{2\triangle t} = \frac{u_{i+1}^n - 2u_i^n + u_{i-1}^n}{(\triangle x)^2} \qquad (A)$$

or

$$\frac{u_i^{n+1} - u_i^{n-1}}{2\triangle t} = \frac{u_{i+1}^n - (u_i^{n+1} + u_i^{n-1}) + u_{i-1}^n}{(\triangle x)^2} \qquad (B)$$

Describe the stability and convergence properties of both methods.