Math 269B, Winter 2003
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Homework #6
Due date: Friday, March 14, 2003

[**1**] Consider the equation

$$u_t = b_{11} u_{xx} + 2b_{12} u_{xy} + b_{22} u_{yy}.$$

(a) Under what conditions is this a parabolic equation ?
(b) Derive a formula for the solution of this equation, using the Fourier transform in two spatial dimensions, function of $\hat{u}_0(\omega_1, \omega_2)$.

[**2**] Consider the convection-diffusion equation $u_t + a u_x = b u_{xx}$, with $b > 0$. Let $y = x - at$ and set $w(t, y) = u(t, y + at)$. Find an equation satisfied by w.

[**3**] Show that a scheme for $u_t = b u_{xx}$ of the form

$$v_m^{n+1} = \alpha v_m^n + \frac{1-\alpha}{2}(v_{m+1}^n + v_{m-1}^n),$$

with α constant as h and k tend to zero, is consistent with the heat equation only if $\alpha = 1 - 2b\mu$.

[**4**] Show that the Crank-Nicholson scheme for $u_t = b u_{xx}$ satisfies the *maximum norm stability*

$$\|v^{n+1}\|_\infty \leq \|v^n\|_\infty$$

for all solutions if $b\mu \leq 1$. *Hint:* Show that if $v_{m'}^{n+1}$ is the largest value of v_m^{n+1}, then

$$v_{m'}^{n+1} \leq -\frac{b\mu}{2} v_{m'-1}^{n+1} + (1 + b\mu) v_{m'}^{n+1} - \frac{b\mu}{2} v_{m'+1}^{n+1} \leq \|v^n\|_\infty.$$

[**5**] Show that the two-dimensional Du Fort-Frankel scheme for the equation $u_t = b(u_{xx} + u_{yy}) + f$ given by

$$\frac{v_{l,m}^{n+1} - v_{l,m}^{n-1}}{2k} = b \frac{v_{l+1,m}^n + v_{l-1,m}^n + v_{l,m+1}^n + v_{l,m-1}^n - 2(v_{l,m}^{n+1} + v_{l,m}^{n-1})}{h^2} + f_m^n,$$

where $\triangle x = \triangle y = h$, is unconditionally stable.