How to find the location of roots of amplification polynomials: Schur and von Neumann polynomials

Let $\phi(z)$ be a polynomial of degree $d$:

$$
\phi(z) = a_d z^d + \ldots + a_0 = \sum_{i=0}^{d} a_i z^i.
$$

We say that $\phi$ is of exact degree $d$ if $a_d$ is not zero.

**Definition 1.** The polynomial $\phi$ is a Schur polynomial if all its roots, $r_\nu$, satisfy

$$ |r_\nu| < 1. $$

**Definition 2.** The polynomial $\phi$ is a von Neumann polynomial if all its roots, $r_\nu$, satisfy

$$ |r_\nu| \leq 1. $$

**Definition 3.** The polynomial $\phi$ is a simple von Neumann polynomial if $\phi$ is a von Neumann polynomial and its roots on the unit circle are simple roots.

**Definition 4.** The polynomial $\phi$ is a conservative polynomial if all its roots lie on the unit circle, i.e. $|r_\nu| = 1$ for all roots $r_\nu$.

For a polynomial $\phi$, we define a polynomial $\phi^*$ by

$$
\phi^*(z) = \sum_{l=0}^{d} \bar{a}_{d-l} z^l,
$$

where the bar on the coefficients of $\phi$ denotes complex conjugate. Note that

$$
\phi^*(z) = \overline{\phi(z-1)} z^d.
$$

Finally, for a polynomial $\phi_0$ we define recursively the polynomial

$$
\phi_{j+1}(z) = \frac{\phi_j^*(0)\phi_j(z) - \phi_j(0)\phi_j^*(z)}{z}.
$$

It is easy to see that the degree of $\phi_{j+1}$ is less than that of $\phi_j$.

**Theorem 1.** $\phi_j$ is a Schur polynomial of exact degree $d$ if and only if $\phi_{j+1}$ is a Schur polynomial of exact degree $d - 1$ and $|\phi_j(0)| < |\phi_j^*(0)|$. 

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Theorem 2. \( \phi_j \) is a simple von Neumann polynomial if and only if either \( |\phi_j(0)| < |\phi_j^*(0)| \) and \( \phi_{j+1} \) is a simple von Neumann polynomial or \( \phi_{j+1} \) is identically zero and \( \phi_j' \) is a Schur polynomial.

Theorem 3. \( \phi_j \) is a von Neumann polynomial of degree \( d \) if and only if either \( \phi_{j+1} \) is a von Neumann polynomial of degree \( d-1 \) and \( |\phi_j(0)| < |\phi_j^*(0)| \) or \( \phi_{j+1} \) is identically zero and \( \phi_j' \) is a von Neumann polynomial.

Theorem 4. \( \phi_j \) is a conservative polynomial if and only if \( \phi_{j+1} \) is identically zero and \( \phi_j' \) is a von Neumann polynomial.