How to find the location of roots of amplification polynomials: Schur and von Neumann polynomials

Let $\phi(z)$ be a polynomial of degree d:

$$\phi(z) = a_d z^d + ... + a_0 = \sum_{l=0}^{d} a_l z^l.$$

We say that ϕ is of exact degree d if a_d is not zero.

Definition 1. *The polynomial ϕ is a Schur polynomial if all its roots, r_ν, satisfy*

$$|r_\nu| < 1.$$

Definition 2. *The polynomial ϕ is a von Neumann polynomial if all its roots, r_ν, satisfy*

$$|r_\nu| \leq 1.$$

Definition 3. *The polynomial ϕ is a simple von Neumann polynomial if ϕ is a von Neumann polynomial and its roots on the unit circle are simple roots.*

Definition 4. *The polynomial ϕ is a conservative polynomial if all its roots lie on the unit circle, i.e. $|r_\nu| = 1$ for all roots r_ν.*

For a polynomial ϕ, we define a polynomial ϕ^* by

$$\phi^*(z) = \sum_{l=0}^{d} \bar{a}_{d-l} z^l,$$

where the bar on the coefficients of ϕ denotes complex conjugate. Note that

$$\phi^*(z) = \overline{\phi(\bar{z}^{-1})} z^d.$$

Finally, for a polynomial ϕ_0 we define recursively the polynomial

$$\phi_{j+1}(z) = \frac{\phi_j^*(0)\phi_j(z) - \phi_j(0)\phi_j^*(z)}{z}.$$

It is easy to see that the degree of ϕ_{j+1} is less than that of ϕ_j.

Theorem 1. *ϕ_j is a Schur polynomial of exact degree d if and only if ϕ_{j+1} is a Schur polynomial of exact degree $d-1$ and $|\phi_j(0)| < |\phi_j^*(0)|$.*

Theorem 2. *ϕ_j is a simple von Neumann polynomial if and only if either $|\phi_j(0)| < |\phi_j^*(0)|$ and ϕ_{j+1} is a simple von Neumann polynomial or ϕ_{j+1} is identically zero and ϕ_j' is a Schur polynomial.*

Theorem 3. *ϕ_j is a von Neumann polynomial of degree d if and only if either ϕ_{j+1} is a von Neumann polynomial of degree $d-1$ and $|\phi_j(0)| < |\phi_j^*(0)|$ or ϕ_{j+1} is identically zero and ϕ_j' is a von Neumann polynomial.*

Theorem 4. *ϕ_j is a conservative polynomial if and only if ϕ_{j+1} is identically zero and ϕ_j' is a von Neumann polynomial.*