

Math 269B, Winter 2003

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Homework #5

Due date: Monday, March 3, 2003

[1] Consider the one-way wave equation

$$\frac{\partial u}{\partial t} + \frac{\partial u}{\partial x} = 0.$$

Analyze the stability and the order of accuracy of the following *angled derivative method*

$$u_j^{n+2} = (1 - 2\lambda)(u_j^{n+1} - u_{j-1}^{n+1}) + u_{j-1}^n, \quad n \geq 0,$$

with $\lambda = \frac{\Delta t}{\Delta x}$.

[2] *Computational assignment.* Consider the one-way wave equation

$$\begin{aligned} \frac{\partial u}{\partial t} + \frac{\partial u}{\partial x} &= 0, \quad t > 0, \quad 0 < x < 2, \\ u(x, 0) &= e^{-100(x-1/2)^2} \sin(20\pi x), \\ u(0, t) = u(2, t) &= 0, \quad t \geq 0. \end{aligned}$$

Find approximations of the exact solution at $t = 1$, using $\Delta x = \frac{1}{80}$ and $\lambda = \frac{\Delta t}{\Delta x} = \frac{2}{3}$, by three methods: the leapfrog method, the Lax-Wendroff method, and the angled derivative method.

Comment and compare the obtained approximations, function of accuracy, dissipation, dispersion, conservation (a non-dissipative scheme is called conservative). Give and plot the exact solution also at $t = 1$.

[3] *Computational assignment.* Consider now the same equation as in [2], but with the initial condition

$$u(x, 0) = \begin{cases} 1, & \text{if } \frac{1}{4} \leq x < \frac{3}{4}, \\ 0, & \text{otherwise.} \end{cases}$$

Find numerical approximations using four methods: the three methods used in [2] and the forward-in-time backward-in-space method (FTBS). In all

cases, take $\Delta x = \frac{1}{100}$ and $\lambda = \frac{\Delta t}{\Delta x} = \frac{2}{3}$. The boundary conditions are as in [2].

Plot the results at $t = 1$, together with the exact solution. Comment and compare the obtained results. Note that here the FTBS scheme is the upwind scheme, in the case $a = 1 > 0$ (positive speed of propagation).

[4] Show that the scheme

$$\frac{v_{l,m}^{n+1} - \frac{1}{4}(v_{l+1,m+1}^n + v_{l-1,m+1}^n + v_{l+1,m-1}^n + v_{l-1,m-1}^n)}{k} + a\Delta_0^x v_{l,m}^n + b\Delta_0^y v_{l,m}^n = 0$$

for the equation $u_t + au_x + bu_y = 0$, with $\Delta x = \Delta y = h$, is stable if and only if $(|a| + |b|)\lambda \leq 1$.

[5] Consider the system of equations

$$\begin{pmatrix} u \\ v \end{pmatrix}_t = A \begin{pmatrix} u \\ v \end{pmatrix}_x; \quad A = \begin{pmatrix} 1 & \alpha \\ 0 & 1 \end{pmatrix},$$

which is approximated by the Lax-Friedrichs scheme, $V_j^n \approx (u(x_j, t_n), v(x_j, t_n))$,

$$V_j^{n+1} = \frac{1}{2}(V_{j+1}^n + V_{j-1}^n) + \frac{\Delta t}{2\Delta x} A(V_{j+1}^n - V_{j-1}^n).$$

Determine the stability properties depending on whether $\alpha \neq 0$ or $\alpha = 0$.

Hint: If $\alpha = 0$ this becomes an uncoupled system of two scalar one-way wave equations, and the stability condition of the scalar Lax-Friedrichs scheme applies for each equation. If $\alpha \neq 0$, show that the scheme is unstable (you can compute the amplification matrix $G(\theta)$, for constant λ , and show that the matrices G^n are unbounded in some matrix norm, for at least one value of θ). Compute an explicit form of G^n .