Practice Problems

[1] Consider the numerical method for solving $y' = f(x, y)$,

$$y_{i+1} = y_i + \frac{h}{2}\Big[f(x_i, y_i) + f(x_{i+1}, y_{i+1})\Big] + \frac{h^2}{12}\Big[y_i'' - y_{i+1}''\Big],$$

where $y_i'' = \frac{\partial f}{\partial x}(x_i, y_i) + f(x_i, y_i)\frac{\partial f}{\partial y}(x_i, y_i)$.

(a) Find the order of the method.

(b) What is the recurrence formula, when this method is applied to the (IVP) $y' = -2y$, $y(0) = 1$?

[2] Consider the two-step method

$$y_{i+1} = \frac{1}{2}(y_i + y_{i-1}) + \frac{h}{4}\Big[4f(x_{i+1}, y_{i+1}) - f(x_i, y_i) + 3f(x_{i-1}, y_{i-1})\Big].$$

(a) What is the order of this method ? Show your work.

(b) Does this method converge ? Explain.

[3] Consider the Runge-Kutta method for solving $y' = F(y)$

$$y_{i+1} = y_i + ahF(y_i) + bhF(y_i + chF(y_i)).$$

Find the coefficients a, b and c, so that the method is of order 2.

[4] Consider the Euler's method applied to $y' = f(x, y)$,

$$y_{i+1} = y_i + hf(x_i, y_i).$$

Let $e_i = y_i - y(x_i)$. Assume that the function $f : [a, b] \times R \to R$ and its first order partial derivatives are continuous and bounded, and that f is Lipschitz with respect to y, with Lipschitz constant M. Assume also that y'' exists in $[a, b]$ and that it is bounded.

Show the inequality:

$$|e_{i+1}| \le (1 + hM)|e_i| + O(h^2).$$