Practice Problems

[1] Consider the numerical method for solving \( y' = f(x, y) \),

\[
y_{i+1} = y_i + \frac{h}{2} \left[ f(x_i, y_i) + f(x_{i+1}, y_{i+1}) \right] + \frac{h^2}{12} \left[ y''_i - y''_{i+1} \right],
\]

where \( y''_i = \frac{\partial f}{\partial x}(x_i, y_i) + f(x_i, y_i) \frac{\partial f}{\partial y}(x_i, y_i). \)

(a) Find the order of the method.

(b) What is the recurrence formula, when this method is applied to the (IVP) \( y' = -2y, \ y(0) = 1 \)?

[2] Consider the two-step method

\[
y_{i+1} = \frac{1}{2} (y_i + y_{i-1}) + \frac{h}{4} \left[ 4f(x_{i+1}, y_{i+1}) - f(x_i, y_i) + 3f(x_{i-1}, y_{i-1}) \right].
\]

(a) What is the order of this method? Show your work.

(b) Does this method converge? Explain.

[3] Consider the Runge-Kutta method for solving \( y' = F(y) \)

\[
y_{i+1} = y_i + ahF(y_i) + bhF(y_i + chF(y_i)).
\]

Find the coefficients \( a, b \) and \( c \), so that the method is of order 2.

[4] Consider the Euler’s method applied to \( y' = f(x, y) \),

\[
y_{i+1} = y_i + hf(x_i, y_i).
\]

Let \( e_i = y_i - y(x_i) \). Assume that the function \( f : [a, b] \times \mathbb{R} \to \mathbb{R} \) and its first order partial derivatives are continuous and bounded, and that \( f \) is Lipschitz with respect to \( y \), with Lipschitz constant \( M \). Assume also that \( y'' \) exists in \([a, b]\) and that it is bounded.

Show the inequality:

\[
|e_{i+1}| \leq (1 + hM)|e_i| + O(h^2).
\]