Math 269A
Sample Practice Problems and Topics for the Final

[**1**] Consider an ODE of the form

$$\text{(IVP)} \quad y' = f(x, y), \quad y(x_0) = y_0, \quad x, x_0 \in [a, b]$$

whose solution is to be approximated using a general one-step method of the form

$$y_{i+1} = y_i + h\Phi(x_i, y_i, h). \tag{1}$$

Assume that the method is of order p so that the local truncation error gives

$$y(x_{i+1}) = y(x_i) + h\Phi(x_i, y(x_i), h) + \tau_i h^{p+1},$$

where $y(x)$ is the exact solution of the (IVP) and τ_i is a constant that depends on derivatives of this solution.

(a) Derive an error estimate for the obtained approximation of the form

$$|e_i| \leq C_1|e_0| + C_2 h^p, \quad i = 1, 2, ..., N$$

where C_1 and C_2 are constants, and $h = \frac{b-a}{N}$. Please state your assumptions concerning Φ.

(b) For a second order Runge-Kutta method of your choice, give the explicit representation of $\Phi(x_i, y_i, h)$ that arises when the method is expressed in form (1).

[**2**] Consider the second order differential equation

$$y'' - 21y' + 20y = 0.$$

(a) Give an equivalent first order system.

(b) Give the stability stepsize restriction if 2nd order Runge-Kutta is used to compute solutions to the first order system.

[**3**] Consider the multi-step method

$$y_{i+4} - y_i + \alpha[y_{i+3} - y_{i+1}] = h[\beta(f_{i+3} - f_{i+1}) + \gamma f_{i+2}],$$

where $f_i = f(x_i, y_i)$.

(a) Determine α, β, and γ so that the method has order 3.

(b) Is this a convergent method ? Explain.

[**4**] Consider the 2nd and 3rd order Runge-Kutta methods
$k_1 = hf(x_i, y_i)$
$k_2 = hf(x_i + \frac{1}{2}h, y_i + \frac{1}{2}k_1)$
$k_3 = hf(x_i + h, y_i - k_1 + 2k_2)$

$y_{i+1} = y_i + k_2$ (2nd order R-K)
$\bar{y}_{i+1} = y_i + \frac{1}{6}k_1 + \frac{2}{3}k_2 + \frac{1}{6}k_3$ (3rd order R-K)

(a) Give the difference equation that results when these methods are applied to the model problem

$$y' = \lambda y.$$

(b) If an adaptive procedure based on this pair of 2nd and 3rd order methods is applied to the model problem, explicitly determine the equation for h_{new} that results when the following formula is used to determine the stepsize:

$$h_{new} = h_{old}\Big(\frac{\epsilon}{|\bar{y}^{i+1} - y^{i+1}|}\Big)^{\frac{1}{p+1}}$$

($p = 2$).

(c) What restriction on the tolerance ϵ is required to ensure that the h_{new} obtained with this formula satisfies the stability restrictions associated with 2nd order Runge-Kutta ?

List of topics discussed in this class:

- Notations and terminology for ODE's and systems of ODE's; reduction of higher order ODE's to 1st order systems of ODE's; the fundamental existence and uniqueness thm. for ODE's (Lipschitz condition).
- Introduction of Euler's method, order of Euler's method, one step methods (introduction, definition, consistency, local truncation error).
- Explicit Runge-Kutta (ERK) methods (introduction of the method in the general case, notations in the general case, derivation of ERK of second order); Runge-Kutta method of fourth order.
- Examples of implicit methods: implicit Euler's method, trapezoidal rule, implicit midpoint rule, the theta method; computation of orders for these methods using the truncation error.
- Convergence of one-step methods (the general case; see also convergence for Euler's method, etc).
- Asymptotic expansions for the global discretization error for one step methods, and applications to error estimate.
- Practical implementation of one step methods
- Linear Multistep methods: examples, derivation using the Lagrange interpolation polynomial
- Linear multistep methods: definition and computations of the local truncation error, order of the method, consistency.
- Implicit and explicit linear multistep methods; predictor-corrector methods.
- Examples of consistent multistep methods which diverge.
- Linear difference equations: stability (root) condition, general solution.
- Convergence Thm. for linear multistep methods
- Order and consistency for linear multistep methods

- Adaptive methods for one-step and multi-step methods, error control, Milne device, extrapolation
- Stiff differential equations, stability and intervals (regions) of absolute stability, A-stable methods, BDF methods
- Numerical methods and stability for systems of ODE's
- Finite difference methods for linear BVP
- Functional (fixed point) iteration and Newton's iteration