Math 269A. HW # 7. Due Friday, December 7.

[**1**] Consider using Euler's method for solving $y' = f(y)$ and advancing the solution from y_i to y_{i+1} twice - once with a stepsize of h and another with a stepsize of $h/2$. The scheme below illustrates this: $y^{(1)}_{i+1}$ is obtained with a step of h, while $y^{(2)}_{i+1}$ is obtained with two steps of size $h/2$.

$$\begin{array}{l} y_i - > y^{(1)}_{i+1} \\ y_i - - - - - - - - - - > - - - - - - - - - - > y^{(2)}_{i+1} \\ x_i - - - - - - -(x_i + h/2) - - - - - - -(x_i + h) \end{array}$$

(a) Let $y^{(*)}_{i+1} = \alpha y^{(1)}_{i+1} + \beta y^{(2)}_{i+1}$. Without using the asymptotic error expansion, how should the values of α and β be chosen so that $y^{(*)}_{i+1}$ is a more accurate solution to the differential equation than either $y^{(1)}_{i+1}$ or $y^{(2)}_{i+1}$? Justify your answer (use local truncation error and Taylor's expansion).

(b) For your choice of α and β what is the order of the local truncation error associated with the scheme that advances the solution using $y^{(*)}_{i+1} = \alpha y^{(1)}_{i+1} + \beta y^{(2)}_{i+1}$?

[**2**] Consider using Euler's method to solve

$$\frac{dy}{dt} = 4t\cos(y), \quad y(0) = 0,$$

up to time $t = 5$.

(a) By estimating the size of $\partial f/\partial y$, estimate the largest time step for which Euler's method is stable.

(b) Use your program from assignment #1 to experimentally determine the largest timestep for which Euler's method is stable. (One can do this by observing the differences between the computed solution and the steady state value $y = \pi/2$).

(c) How does your analytical estimate of the timestep compare to the experimentally determined timestep ?

[**3**] For the numerical solution of the problem

$$y' = \lambda(y - \sin t) + \cos t, \quad y(0) = 1, \quad 0 \le t \ \le 1,$$

whose exact solution is $y(t) = e^{\lambda t} + \sin t$, consider using the following three two-step methods, with $y_0 = 1$ and $y_1 = y(h)$ (i.e., using the exact solution so as not to worry here about y_1).

(a) The mid-point two-step method

$$y_n = y_{n-2} + 2hf(t_{n-1}, y_{n-1}).$$

(b) The Adams-Bashforth method

$$y_n = y_{n-1} + \frac{h}{2}(3f(t_{n-1}, y_{n-1}) - f(t_{n-2}, y_{n-2})).$$

(c) BDF

$$y_n = \frac{4y_{n-1} - y_{n-2}}{3} + \frac{2h}{3} f(t_n, y_n).$$

Consider using $h = 0.01$ for $\lambda = 10$, $\lambda = -10$, and $\lambda = -500$. Discuss the expected quality of the obtained solutions in these nine calculations. Try to do this without calculating any of these solutions. Then confirm your predictions by doing the calculations (the intervals of absolute stability of these three methods have been derived or given in class or in the assignments).

[**4**] The problem

$$\frac{dy}{dt} = \sqrt{y}, \quad y(0) = 0$$

has the nontrivial solution $y(x) = (x/2)^2$. Application of Euler's method however yields $y_j = 0$ for all j and any $h = dt$. Explain this paradox.

[**5**] Consider the 2nd and 3rd order Runge-Kutta methods
$k_1 = hf(x_i, y_i)$
$k_2 = hf(x_i + \frac{1}{2}h, y_i + \frac{1}{2}k_1)$
$k_3 = hf(x_i + h, y_i - k_1 + 2k_2)$

$y_{i+1} = y_i + k_2$ (2nd order R-K)
$\bar{y}_{i+1} = y_i + \frac{1}{6}k_1 + \frac{2}{3}k_2 + \frac{1}{6}k_3$ (3rd order R-K)
(a) Give the difference equation that results when these methods are applied to the model problem

$$y' = \lambda y.$$

(b) If an adaptive procedure based on this pair of 2nd and 3rd order methods is applied to the model problem, explicitly determine the equation for h_{new} that results when the following formula is used to determine the stepsize:

$$h_{new} = h_{old}\Big(\frac{\epsilon}{|\bar{y}^{i+1} - y^{i+1}|}\Big)^{\frac{1}{p+1}}$$

$(p = 2)$.

(c) What restriction on the tolerance ϵ is required to ensure that the h_{new} obtained with this formula satisfies the stability restrictions associated with 2nd order Runge-Kutta ?

[**6**] Consider the two-step method

$$y_{i+1} = \frac{1}{2}(y_i + y_{i-1}) + \frac{h}{4}\Big[4f(x_{i+1}, y_{i+1}) - f(x_i, y_i) + 3f(x_{i-1}, y_{i-1})\Big].$$

(a) What is the order of this method ?

(b) Does this method converge ? Explain.