Math 269A. HW #6. Due: Wednesday, Nov. 28, or Friday, Nov. 30

[**1**]
(a) Give the definition of an A-stable method.
(b) Determine all values of θ such that the theta method given below is A-stable.

$$y_{i+1} = y_i + h\Big[\theta f(x_i, y_i) + (1-\theta) f(x_{i+1}, y_{i+1})\Big], \quad i = 0, 1, ...$$

[**2**] The two-step method $y_{i+1} = y_{i-1} + 2hf(x_i, y_i)$ is called the *explicit midpoint rule.*

(a) Implement this two-step method for the very simple differential equation $y' = -y$, $y(0) = 1$ (the exact solution is e^{-x}). Use $y_1 = y(h) = e^{-h}$ and the values $h = 1/2$, $h = 1/4$, $h = 1/8$, $h = 1/16$. Plot the exact solution and the numerical approximations on the interval [0,8]. You should turn in the code and the plot of values.

(b) Show that the region of absolute stability for the explicit midpoint rule is the empty set.

[**3**] Consider the following (IVP):

$$\begin{aligned} y_1' &= 198y_1 + 199y_2, & y_1(0) = 1 \\ y_2' &= -398y_1 - 399y_2, & y_2(0) = -1, \end{aligned}$$

that we write in matrix-vector form $\vec{y}' = A\vec{y}$, $\vec{y}(0) = \vec{y}_0$.

(a) Find the exact solution of this autonomous linear system. What is its asymptotic behavior, as $x \to \infty$?

(b) Compute the eigenvalues λ_1 and λ_2 of the matrix A and the corresponding matrix P of eigenvectors. What relation exists between A, P and $\Lambda = diag(\lambda_1, \lambda_2)$?

(c) Is this a stiff system of ODE's ? If yes, what is the stiffness ratio ? Explain.

(d) Apply the trapezoidal rule to this system following the steps:

(i) Express $\vec{y}_{j+1}$ function of $\vec{y}_j$, by a recurrence formula given in matrix-vector form.

(ii) If $\vec{z}_j$ is defined such that $\vec{y}_j = P\vec{z}_j$, express $\vec{z}_{j+1}$ function of $\vec{z}_j$ in matrix-vector form and the associated scalar recursions for each component of $\vec{z}_j$.

(e) If the system is solved using the trapezoidal method, what restriction, if any, has to be imposed on the stepsize h to obtain a correct qualitative behavior ?

[**4**] (a) Give the solution (e.g. explicit formulas for $y_1(x)$ and $y_2(x)$) to

$$\frac{\partial}{\partial x}\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \begin{pmatrix} -2 & -2 \\ 2 & -2 \end{pmatrix}\begin{pmatrix} y_1 \\ y_2 \end{pmatrix}, \qquad \begin{pmatrix} y_1(0) \\ y_2(0) \end{pmatrix} = \begin{pmatrix} -\sqrt{2} \\ \sqrt{2} \end{pmatrix}.$$

Give an estimate of the stepsize required to obtain a qualitatively correct solution if one is using Euler's method.

[**5**] Consider the second order differential equation

$$y'' + 19y' - 20y = 0.$$

(a) Give an equivalent first order system for this equation.

(b) Give the stability stepsize restriction if backward Euler is used to compute solutions to the first order system.