MATH 269A HW #3 (due on Friday, October 26)

[1] Consider $\Phi = \Phi(x, y, h)$ associated with the 2nd order R-K methods:

$$\Phi(x, y, h) = \frac{1}{2}\Big[f(x, y) + f(x + h, y + hf(x, y))\Big]$$

$$\Phi(x, y, h) = f(x + \frac{h}{2}, y + \frac{h}{2}f(x, y)).$$

Verify that, under appropriate assumptions on f, we have:

$$\frac{\partial \Phi}{\partial y} = \frac{\partial f}{\partial y} + O(h).$$

(we have used this property in the proof of the asymptotic expansion of the global discretization error).

[2] Assume $f = f(x, y)$ is analytic, Lipschitz with respect to y on some domain, with the Lipschtiz constant M, bounded and with bounded first-order derivatives. Assume that the exact solution $y = y(x)$ of the initial value problem $y'(x) = f(x, y(x))$, $y(x_0) = y_0$ is also analytic. Show that Euler's method is convergent (you can follow some steps from the proof of the general convergence theorem done in class, Thm. 7.2.2.3 from Stoer-Bulirsch; also you can use Lemma 7.2.2.2 done in class). The proof has to be given for this particular case.

[3] Let $y_{x,h}$ be the approximate solution furnished by Euler's method for the initial-value problem $y' = y$, $y(0) = 1$. Show:

(a) $y_{x,h} = (1 + h)^{x/h}$.

(b) In the expansion $y_{x,h} = e_0(x) + e_1(x)h + e_2(x)h^2 + ...$, with $e_0(x) = e^x$, find $e_1(x)$ (use the fact that the exact solution is known).

[4] Consider again the (IVP)

$$y' = \frac{3x^2 + 4x + 2}{2(y - 1)}, \quad y(0) = -1.$$

(a) Implement the classical fourth-order Runge-Kutta method and use it to numerically solve the (IVP) (done in the previous homework, no need to repeat).

(b) Without using the formula of the exact solution, find an approximation of the error $e(x, h) = e(2, 0.05)$ at step $h = 0.05$ for $x = 2$ (use two timesteps $h = 0.1$ and $h = 0.05$, and the application from the end of section 7.2.3, Stoer and Bulirsch, covered in class). Compare this with the actual error (when the exact solution is known).