MATH 269A, Fall 2002. HW #3 (due Friday, October 18)

1. Consider $\Phi = \Phi(x, y, h)$ associated with the 2nd order R-K methods:

$$\Phi(x, y, h) = \frac{1}{2}[f(x, y) + f(x + h, y + hf(x, y))]$$

$$\Phi(x, y, h) = f(x + \frac{h}{2}, y + \frac{h}{2}f(x, y)).$$

Verify that, under appropriate assumptions on f, we have:

$$\frac{\partial \Phi}{\partial y} = \frac{\partial f}{\partial y} + O(h).$$

(we have used this property in general, in the proof of the asymptotic expansion of the global error).

2. Assume $f = f(x, y)$ is analytic, Lipschitz with respect to y on some domain, with the Lipschtiz constant M, bounded and with bounded first-order derivatives. Assume that the exact solution $y = y(x)$ of the initial value problem $y'(x) = f(x, y(x))$, $y(x_0) = y_0$ is also analytic.

Show that Euler's method is convergent (follow some steps from the proof of the general convergence theorem; also use Lemma 7.2.2.2). The proof has to be given for this particular case.

3. Let $y_{x,h}$ be the approximate solution furnished by Euler's method for the initial-value problem $y' = y$, $y(0) = 1$. Show:

(a) $y_{x,h} = (1 + h)^{x/h}$.

(b) In the expansion $y_{x,h} = e_0(x) + e_1(x)h + e_2(x)h^2 + ...$, with $e_0(x) = e^x$, find $e_1(x)$ (use the fact that the exact solution is known).

4. Consider again the (IVP)

$$y' = \frac{3x^2 + 4x + 2}{2(y - 1)}, \quad y(0) = -1.$$

(a) Implement the classical fourth-order Runge-Kutta method and use it to numerically solve the (IVP).

(b) Without using the formula of the exact solution, find an approximation of the error $e(x, h) = e(2, 0.05)$ at step $h = 0.05$ for $x = 2$ (use two timesteps $h = 0.1$ and $h = 0.05$, and the application from the end of section 7.2.3, Stoer and Bulirsch). Compare this with the actual error (when the exact solution is known).