MATH 269A, Fall 2002. HW #3 (due Friday, October 18)
1. Consider \( \Phi = \Phi(x, y, h) \) associated with the 2nd order R-K methods:
\[
\Phi(x, y, h) = \frac{1}{2}[f(x, y) + f(x + h, y + hf(x, y))]
\]
\[
\Phi(x, y, h) = f(x + \frac{h}{2}, y + \frac{h}{2}f(x, y)).
\]
Verify that, under appropriate assumptions on \( f \), we have:
\[
\frac{\partial \Phi}{\partial y} = \frac{\partial f}{\partial y} + O(h).
\]
(we have used this property in general, in the proof of the asymptotic expansion of the global error).

2. Assume \( f = f(x, y) \) is analytic, Lipschitz with respect to \( y \) on some domain, with the Lipschitz constant \( M \), bounded and with bounded first-order derivatives. Assume that the exact solution \( y = y(x) \) of the initial value problem \( y'(x) = f(x, y(x)) \), \( y(x_0) = y_0 \) is also analytic.

Show that Euler’s method is convergent (follow some steps from the proof of the general convergence theorem; also use Lemma 7.2.2.2). The proof has to be given for this particular case.

3. Let \( y_{x,h} \) be the approximate solution furnished by Euler’s method for the initial-value problem \( y' = y, y(0) = 1 \). Show:
   (a) \( y_{x,h} = (1 + h)^{x/h} \).
   (b) In the expansion \( y_{x,h} = e_0(x) + e_1(x)h + e_2(x)h^2 + \ldots \), with \( e_0(x) = e^x \), find \( e_1(x) \) (use the fact that the exact solution is known).

4. Consider again the (IVP)
\[
y' = \frac{3x^2 + 4x + 2}{2(y - 1)}, \quad y(0) = -1.
\]
   (a) Implement the classical fourth-order Runge-Kutta method and use it to numerically solve the (IVP).
   (b) Without using the formula of the exact solution, find an approximation of the error \( e(x, h) = e(2, 0.05) \) at step \( h = 0.05 \) for \( x = 2 \) (use two timesteps \( h = 0.1 \) and \( h = 0.05 \), and the application from the end of section 7.2.3, Stoer and Bulirsch). Compare this with the actual error (when the exact solution is known).