Sobolev spaces: properties

**Theorem:** (Leibniz’s formula) Assume \( u \in W^{m,p}(\Omega) \), \(|\alpha| \leq m\). If \( \xi \in C_0^\infty(\Omega) \), then \( \xi u \in W^{m,p}(\Omega) \) and

\[
D^\alpha (\xi u) = \sum_{\beta \leq \alpha} C_\beta^\alpha D^\beta \xi D^{\alpha - \beta} u,
\]

where \( C_\beta^\alpha = \frac{\alpha!}{\beta!(\alpha - \beta)!} \), \( \alpha! = \alpha_1! \alpha_2! \cdots \alpha_n! \) for \( \alpha = (\alpha_1, \alpha_2, \ldots, \alpha_n) \).

**Transformation of coordinates** Let \( \Phi : \Omega \to G \) be a 1-to-1 and onto transformation, with inverse \( \Psi = \Phi^{-1} \), in \( n \) dimensions. We assume \( \Phi \in C^m(\Omega)^n \) and \( \Psi \in C^m(G)^n \). There are constants \( 0 < c \leq C \) s.t. \( c \leq |\det \nabla \Phi(x)| \leq C \) for all \( x \in \Omega \). Using the notation \( y = \Phi(x) \), we define for a measurable function \( u \) on \( \Omega \), the measurable function \( Au \) on \( G \) by \( Au(y) := u(\Psi(y)) \).

**Theorem:** \( A \) transforms \( W^{m,p}(\Omega) \) boundedly onto \( W^{m,p}(G) \), and has a bounded inverse. In other words, there are constants \( C_1, C_2 \) s.t.

\[
C_1 \| u \|_{m,p,\Omega} \leq \| Au \|_{m,p,G} \leq C_2 \| u \|_{m,p,\Omega},
\]

for all \( u \in W^{m,p}(\Omega) \).

**Particular case of Rellich-Kondrachov Theorem** Assume \( \Omega \) open, bounded, and \( \partial \Omega \) Lipschitz, \( 1 \leq p \leq \infty \). Then the canonical embedding \( W^{1,p}(\Omega) \to L^p(\Omega) \) is compact. In other words, we have:

(i) There is a constant \( C \) such that \( \| u \|_{L^p(\Omega)} \leq C \| u \|_{W^{1,p}(\Omega)} \), for all \( u \in W^{1,p}(\Omega) \).

(ii) If \( \{ u_n \} \) is a bounded sequence in \( W^{1,p}(\Omega) \), then there is a subsequence \( \{ u_{n_j} \} \) of \( \{ u_n \} \) convergent in \( L^p(\Omega) \).

**Remark:** Please note that under the same assumptions on \( \Omega \), as a corollary, if \( 1 < p < \infty \), if \( \{ u_n \} \) is a bounded sequence in \( W^{1,p}(\Omega) \), then there is a subsequence \( \{ u_{n_j} \} \) and \( u \in W^{1,p}(\Omega) \) s.t. \( u_{n_j} \) converges to \( u \) strongly in \( L^p(\Omega) \) and weakly in \( W^{1,p}(\Omega) \).