Sobolev spaces: properties

Theorem: (Leibniz's formula) Assume $u \in W^{m,p}(\Omega)$, $|\alpha| \leq m$. If $\xi \in C_0^\infty(\Omega)$, then $\xi u \in W^{m,p}(\Omega)$ and

$$D^\alpha(\xi u) = \sum_{\beta \leq \alpha} C_\beta^\alpha D^\beta \xi D^{\alpha-\beta} u,$$

where $C_\beta^\alpha = \frac{\alpha!}{\beta!(\alpha-\beta)!}$, $\alpha! = \alpha_1!\alpha_2!...\alpha_n$ for $\alpha = (\alpha_1, \alpha_2, ..., \alpha_n)$.

Transformation of coordinates Let $\Phi : \Omega \to G$ be a 1-to-1 and onto transformation, with inverse $\Psi = \Phi^{-1}$, in n dimensions. We assume $\Phi \in C^m(\overline{\Omega})^n$ and $\Psi \in C^m(\overline{G})^n$. There are constants $0 < c \leq C$ s.t. $c \leq |\det \nabla\Phi(x)| \leq C$ for all $x \in \Omega$. Using the notation $y = \Phi(x)$, we define for a measurable function u on Ω, the measurable function Au on G by $Au(y) := u(\Psi(y))$.

Theorem: A transforms $W^{m,p}(\overline{\Omega})$ boundedly onto $W^{m,p}(G)$, and has a bounded inverse. In other words, there are constants C_1, C_2 s.t.

$$C_1\|u\|_{m,p,\Omega} \leq \|Au\|_{m,p,G} \leq C_2\|u\|_{m,p,\Omega},$$

for all $u \in W^{m,p}(\Omega)$.

Particular case of Rellich-Kondrachov Theorem Assume Ω open, bounded, and $\partial\Omega$ Lipschitz, $1 \leq p \leq \infty$. Then the canonical embedding $W^{1,p}(\Omega) \to L^p(\Omega)$ is compact. In other words, we have:

(i) There is a constant C such that $\|u\|_{L^p(\Omega)} \leq C\|u\|_{W^{1,p}(\Omega)}$, for all $u \in W^{1,p}(\Omega)$.

(ii) If $\{u_n\}$ is a bounded sequence in $W^{1,p}(\Omega)$, then there is a subsequence $\{u_{n_j}\}$ of $\{u_n\}$ convergent in $L^p(\Omega)$.

Remark: Please note that under the same assumptions on Ω, as a corollary, if $1 < p < \infty$, if $\{u_n\}$ is a bounded sequence in $W^{1,p}(\Omega)$, then there is a subsequence $\{u_{n_j}\}$ and $u \in W^{1,p}(\Omega)$ s.t. u_{n_j} converges to u strongly in $L^p(\Omega)$ and weakly in $W^{1,p}(\Omega)$.
