HW #6 Math 265B, L. Vese

Due no later than December 12, 2008. After Dec. 5, you can leave your homework with Babette Dalton (MS 7619, each day before 3pm); or you can slide it under the door of my office, MS 7620-D, or send me an e-mail to let me know where I can find it. If you are interested in reading additional material, not presented in class, please let me know. The latex file of this homework can be found on the class webpage.

[1] *The Nonhomogeneous Dirichlet problem* is

$$-\triangle u = f \text{ in } \Omega, \quad u = g_0 \text{ on } \partial\Omega,$$

for some functions $g_0 : \partial\Omega \to R$, $f : \Omega \to R$.

Let Ω be open, bounded, regular connected set. Assume that there is a function $g \in H^1(\Omega)$ s.t. $Trace(g)_{|\partial\Omega} = g_0$, $f \in L^2(\Omega)$. Define $C = \{v \in H^1(\Omega) : \ Trace(v) = g_0 \text{ on } \partial\Omega\}$. Show that:

(i) C is a closed convex nonempty set of $H^1(\Omega)$. Give another definition of C using $H_0^1(\Omega)$.

(ii) There is a unique solution to the minimization problem

$$\min_{v\in C} \frac{1}{2}\int_\Omega |\nabla v|^2 x - \int_\Omega f v dx.$$

(iii) The solution u from (ii) is characterized by

$$u \in C, \quad \int_\Omega \nabla u \cdot \nabla v dx = \int_\Omega f v dx, \quad \forall v \in H_0^1(\Omega).$$

(iv) Transform the problem in an equivalent way by substitution, so that the Lax-Milgram theorem can be applied, and verify the assumptions.

[2] *Application of $H^s(R^n)$ spaces* Let $s \in R$. Show that for all $f \in H^s(R^n)$, there is a unique u in $H^{s+2}(R^n)$, solution of

$$(-\triangle + k^2)u = f \text{ in } \mathcal{D}'(R^n),$$

with $k \in R$, $k \neq 0$. For what values of s and n is the solution u a classical solution ?

[3] *The semicoercive homogeneous Neumann problem* is: given $f \in L^2(\Omega)$, find $u : \overline{\Omega} \to R$ solution of

$$-\triangle u = f \text{ in } \Omega, \quad \frac{\partial u}{\partial \vec{n}} = 0 \text{ on } \partial\Omega,$$

with Ω open, bounded, connected and with $\partial\Omega$ Lipschitz. Note that if u solution, then $u + const.$ is another solution.

(i) If u is a classical solution, show that $\int_\Omega f(x)dx = 0$.

(ii) Assume that $\int_\Omega f(x)dx = 0$. Define a closed subspace V of $H^1(\Omega)$ s.t. the weak variational formulation of the problem has a unique solution in V based on Lax-Milgram.

(iii) Assume $\int_\Omega f(x)dx = 0$ and consider the regularized problem for any $\epsilon > 0$: let $u_\epsilon \in H^1(\Omega)$ be the unique solution of

$$\int_\Omega (\epsilon u_\epsilon v + \nabla u_\epsilon \cdot \nabla v)dx = \int_\Omega f v dx, \quad \forall v \in H^1(\Omega).$$

(we know by now that this problem has a unique solution based on L-M). Show that u_ϵ norm converges in $H^1(\Omega)$, as $\epsilon \to 0$, to the solution u from (ii).

[4] *The nonlinear Laplacian $\triangle_p$* Let $1 < p < \infty$, Ω open and bounded, $f \in L^\infty(\Omega)$.

(i) Show that there is a unique solution $u \in W_0^{1,p}(\Omega)$ of the minimization

$$\min_{v\in W_0^{1,p}(\Omega)} J(v) = \frac{1}{p}\int_\Omega |\nabla v|^p dx - \int_\Omega f v dx$$

(since we are on a reflexive Banach space, it is sufficient to show that $J : W_0^{1,p}(\Omega) \to 0$ is convex, continuous and coercive)

(ii) Give the equivalent weak variational formulation satisfied by the solution u.