**HW #5** Math 265B, L. Vese
Due on Monday, Dec. 1st

[**1**] Let $u \in W^{m,p}(\Omega)$, $|\alpha| \leq m$. Then $D^\beta(D^\alpha u) = D^\alpha(D^\beta u) = D^{\alpha+\beta} u$ for all multiindices $\alpha, \beta$ with $|\alpha| + |\beta| \leq m$.

[**2**] Let $\Omega = \{x \in R^2 : \ |x_1| < 1, |x_2| < 1\}$. Define

$$u(x) = \begin{cases} 1 - x_1 \text{ if } x_1 > 0, \ |x_2| < x_1 \\ 1 + x_1 \text{ if } x_1 < 0, \ |x_2| < -x_1 \\ 1 - x_2 \text{ if } x_2 > 0, \ |x_1| < x_2 \\ 1 + x_2 \text{ if } x_2 < 0, \ |x_1| < -x_2. \end{cases}$$

For which $1 \leq p \leq \infty$ does $u$ belong to $W^{1,p}(\Omega)$ ?

[**3**] (Weak variational formulation of the homogeneous Dirichlet problem) Let $f \in L^2(\Omega)$, with $\Omega$ open and bounded.

(i) Show that there exists a unique $u \in H_0^1(\Omega)$ such that

$$\int_\Omega \nabla u \cdot \nabla v dx = \int_\Omega f v dx \text{ for all } v \in H_0^1(\Omega).$$

(ii) Give the equivalent minimization formulation.

[**4**] (Weak variational formulation of the coercive homogeneous Neumann problem) Let $a_0 \in L^\infty(\Omega)$, with $\Omega$ open, bounded, $\partial\Omega$ regular, $f \in L^2(\Omega)$. Assume that there is $\alpha_0 > 0$ s.t.

$$a_0(x) \geq \alpha_0 \text{ for a.e. } x \in \Omega.$$

(i) Show that there is a unique solution $u \in H^1(\Omega)$ such that

$$\int_\Omega (\nabla u \cdot \nabla v + a_0 u v) dx = \int_\Omega f v dx \text{ for all } v \in H^1(\Omega).$$

(ii) Give the equivalent minimization formulation.

(iii) Using the following example, show that $\{v \in H^1(\Omega) : \frac{\partial v}{\partial \vec{n}} = 0 \text{ on } \partial\Omega\}$ is not closed in the $H^1(\Omega)$ topology: take $\Omega = (0,1)$, $v(x) = x$, and

$$v_n(x) = \begin{cases} \frac{1}{n} & \text{if } \ 0 \leq x \leq \frac{1}{n}, \\ x & \text{if } \ \frac{1}{n} \leq x \leq 1 - \frac{1}{n}, \\ 1 - \frac{1}{n} & \text{if } \ 1 - \frac{1}{n} \leq x \leq 1. \end{cases}$$

(Thus, the space $H^1(\Omega)$ is the most natural choice to solve the homogeneous Neumann problem, and not a subset of it.)