HW #4 Math 265B, L. Vese
Due on Wednesday, November 19

[**1**] Suppose f continuously differentiable on (a, b) except at $x_1, ..., x_m$, where f has jump discontinuities (with $a < x_1 < ... < x_m < b$), and that its pointwise derivative $\frac{\partial f}{\partial x}$ (defined except at the x_j's) is in $L^1(a, b)$. Show that the distribution derivative f' of f is given by

$$f' = \frac{\partial f}{\partial x} + \sum_{j=1}^{m} \Big[f(x_j+) - f(x_j-)\Big]\tau_{x_j}\delta,$$

where τ_{x_j} is the translation operator.

[**2**] In one dimension, let $\Omega = (0, 2)$ and $u(x) = \begin{cases} x & \text{if} \quad 0 < x \leq 1, \\ 1 & \text{if} \quad 1 < x < 2. \end{cases}$
Show that $u \in W^{1,\infty}(0, 2)$.

[**3**] In one dimension, take $\Omega = (0, 2)$ and $u(x) = \begin{cases} x & \text{if} \quad 0 < x \leq 1, \\ 2 & \text{if} \quad 1 < x < 2. \end{cases}$
Show that u' does not exist in the weak (distributional) sense.

[**4**] In two dimensions, let $\Omega = B(0, 1)$, the open unit ball, and $u(x) = |x|^{-\alpha}$ (for $x \in \Omega \setminus \{0\}$). For which values of $\alpha > 0$ and $p \geq 1$ does u belong to $W^{1,p}(\Omega)$? (since u is smooth away from 0, apply integration by parts on $\Omega \setminus B(0, \epsilon)$ and take $\epsilon \to 0$).

[**5**] In two dimensions, take $\Omega = B(0, R)$ the open ball of radius $R < 1$, and the function $v(x) = \Big|\ln|x|\Big|^k$, where k is a real parameter.

Show that $u \in H^1(\Omega)$ iff $2 - 2k > 1$, i.e. $k < \frac{1}{2}$ (use radial coordinates $|x| = r$ in $\int_\Omega (v^2(x) + |\nabla v(x)|^2) dx$ and make a change of variable $t = -\ln r$ to obtain the answer).

Remark: We see by the above examples that, in 1D, a Sobolev function must be equal a.e. with a continuous function, thus it must have a continuous representative; however, this is no longer true in more than one dimension (e.g., Sobolev functions in nD with $n > 1$ may blow up at some point and do not have a continuous representative).