HW #3 Math 265B, L. Vese
Due on Friday, November 7

[1] Let Ω be an open subset of R^n.
(a) Recall the definition of $\|\cdot\|_\infty$ for measurable functions on Ω
(b) Show that $\|\cdot\|_\infty$ is a norm on the space $L^\infty(\Omega)$
(c) Verify Hölder's inequality for the case $p = 1$, $p' = \infty$ (or $p = \infty$, $p' = 1$).

[2] If $1 \leq p < r \leq \infty$, then $L^p \cap L^r$ is a Banach space with norm $\|u\| = \|u\|_p + \|u\|_r$.

[3] If $1 \leq p < r \leq \infty$, then $L^p + L^r$ is a Banach space with norm $\|u\| = \inf\{\|v\|_p + \|w\|_r : \ u = v + w\}$.

[4] If $g \in L^\infty$, the operator T defined by $Tu = ug$ is bounded on L^p for $1 \leq p \leq \infty$. Its operator norm is at most $\|g\|_\infty$.

[5] Consider in $L^2(0,1)$ the sequence $u_n(x) = \sin(2\pi xn)$, for $x \in (0,1)$, as $n \to \infty$. Show that $u_n \rightharpoonup 0$ in $L^2(0,1)$, but we do not have $u_n \to 0$ in $L^2(0,1)$ (we have weak convergence, but not strong convergence).

[6] Prove by induction the general Hölder inequality. Let $1 \leq p_1, ..., p_m \leq \infty$, with $\frac{1}{p_1} + \frac{1}{p_2} + ... + \frac{1}{p_m} = 1$, and assume $u_k \in L^{p_k}(\Omega)$, for $k = 1, ..., m$. Then

$$\int_\Omega |u_1 \cdots u_m| dx \leq \Pi_{k=1}^m \|u_k\|_{L^{p_k}(\Omega)}.$$